Graded limits of finite-dimensional modules over quantum loop algebras

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Introduction

Theorem (Jacobi-Trudi determinant formula)

For a partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$,

$$s_\lambda(x) = \det\left(h_{\lambda_i - i + j}(x)\right)_{1 \leq i,j \leq n}.$$

$s_\lambda(x)$: Schur polynomial, $h_k(x)$: complete symm. polynomial.

Translation in the $\mathfrak{sl}_{n+1}$-modules

$\lambda \in P^+$: dom. int. wt $\rightsquigarrow \lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$ by $\lambda_i = \sum_{k \geq i} \langle h_k, \lambda \rangle$

$\mathrm{ch}\, V(\lambda) = s_\lambda(x)$, $\mathrm{ch}\, V(k\varpi_1) = h_k(x)$ $\quad (V(\lambda)$: simple $\mathfrak{sl}_{n+1}$-mod.)

$$\rightsquigarrow \mathrm{ch}\, V(\lambda) = \det\left(\mathrm{ch}\, V\left((\lambda_i - i + j)\varpi_1\right)\right)_{1 \leq i,j \leq n}.$$

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$$\mathrm{ch}\, V(\lambda) \neq \det\left(\mathrm{ch}\, V((\lambda_i - i + j)\varpi_1)\right)_{1 \leq i,j \leq n},$$

if $\mathfrak{g} \neq \mathfrak{sl}_{n+1}$ (though there may be several generalizations.)

However this does hold in other types, if the $\mathfrak{g}$-modules are replaced by $U_q(\mathcal{L}\mathfrak{g})$-modules! More precicely, we can show that

$$\mathrm{ch}\, L_q(\lambda) = \det\left(\mathrm{ch}\, L_q((\lambda_i - i + j)\varpi_1)\right)_{1 \leq i,j \leq n}$$

for $\mathfrak{g}$ of type $ABCD$, where $L_q(\mu)$ are minimal affinizations (a special class of f.d. simple $U_q(\mathcal{L}\mathfrak{g})$-modules explained later).

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Plan

1. Definition of minimal affinizations $L_q(\lambda)$
2. Main Theorem (JT formula for $\mathrm{ch}\, L_q(\lambda)$)
3. Proof (Combination of results proved by [N], [Chari-Greenstein], [Sam])

In the proof, **graded limits** ($\mathbb{Z}$-graded $\mathfrak{g} \otimes \mathbb{C}[t]$-modules) are used.

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Minimal affinization

$\mathfrak{g}$: simple Lie algebra of rank n,

$\mathcal{L}\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$: loop algebra, $\big([x \otimes f, y \otimes g] = [x, y] \otimes fg\big)$

$U_q(\mathcal{L}\mathfrak{g})$: quantum loop algebra/$\mathbb{C}(q)$ $\big(q$-analog of $U(\mathcal{L}\mathfrak{g})\big)$

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$U_q(\mathfrak{g})$: quantum group assoc. with $\mathfrak{g}$ $\big(q$-analog of $U(\mathfrak{g})\big)$

(Note: $\mathfrak{g} = \mathfrak{g} \otimes 1 \subseteq \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] = \mathcal{L}\mathfrak{g}$)

Fact

(1) $\{$f.d. simple $\mathfrak{g}$-mod.$\} \overset{1:1}{\longleftrightarrow} P^+ \overset{1:1}{\longleftrightarrow} \{$f.d. simple $U_q(\mathfrak{g})$-mod$\}$

$$V(\lambda) \in \{\text{f.d. simple } \mathfrak{g}\text{-mod.}\}, \quad \lambda \in P^+, \quad V_q(\lambda) \in \{\text{f.d. simple } U_q(\mathfrak{g})\text{-mod}\}$$

(2) The cat. of f.d. $\mathfrak{g}$-modules and $U_q(\mathfrak{g})$-modules are semisimple.

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Minimal affinization

Fact. V: an arbitrary f.d. simple $U_q(\mathcal{L}\mathfrak{g})$-module

$\rightsquigarrow {}^{\exists!}\lambda \in P^+$ s.t. $V \cong V_q(\lambda) \oplus \bigoplus_{\mu<\lambda} V_q(\mu)^{\oplus m_\mu(V)}$ as a $U_q(\mathfrak{g})$-module.

In this case, V is called an **affinization** of $V_q(\lambda)$.

$\{U_q(\mathfrak{g})$-isom. classes of affiniz. of $V_q(\lambda)\} \Leftarrow$ partial order is defined

($[V] \geq [W] \Leftrightarrow \{m_\mu(V)\}_\mu \geq \{m_\mu(W)\}_\mu$ w.r.t. lexicographic order)

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Examples of Minimal affinizations

Minimal affinizations for $\mathfrak{g} = \mathfrak{sl}_{n+1}$

When $\mathfrak{g} = \mathfrak{sl}_{n+1}$, ${}^{\exists}$alg. hom. $\varphi \colon U_q(\mathcal{L}\mathfrak{g}) \twoheadrightarrow U_q(\mathfrak{g})$ (evaluation map)

(q-analog of the map $\mathcal{L}\mathfrak{g} \twoheadrightarrow \mathfrak{g} \colon x \otimes f \to f(a)x$ for any $a \in \mathbb{C}^{\times}$)

$\rightsquigarrow \varphi^* V_q(\lambda)$: simple $U_q(\mathcal{L}\mathfrak{g})$-mod. $\Leftarrow$ minimal affinization of $V_q(\lambda)$

($\because \varphi^* V_q(\lambda) \cong V_q(\lambda)$ as a $U_q(\mathfrak{g})$-mod.)

Remark. If $\mathfrak{g} \neq \mathfrak{sl}_{n+1}$, evaluation map **does not** exist.

$\rightsquigarrow$ Most of minimal affinizations are reducible as a $U_q(\mathfrak{g})$-module, and it is not easy to determine the decompositions.

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Main Theorem

In the sequel, assume that $\mathfrak{g}$ is of type $ABCD$.

Let $\lambda \in P^+$, and let $L_q(\lambda)$ be a minimal affinization of $V_q(\lambda)$.

Theorem

Assume that $\begin{cases} \langle h_n, \lambda \rangle = 0 & \text{if } \mathfrak{g}\text{: type } BC, \\ \langle h_{n-1}, \lambda \rangle = \langle h_n, \lambda \rangle = 0 & \text{if } \mathfrak{g}\text{: type } D. \end{cases}$

Then we have

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where $\lambda_i := \sum_{k \geq i} \langle h_i, \lambda \rangle \in \mathbb{Z}_{\geq 0}$ for $1 \leq i \leq n$.

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Comments on the theorem

$$\mathrm{ch}\, L_q(\lambda) = \det\left(\mathrm{ch}\, L_q((\lambda_i - i + j)\varpi_1)\right)_{1\le i,j\le n}.$$

1. In type A, this is the JT formula since $\mathrm{ch}\, L_q(\lambda) = \mathrm{ch}\, V(\lambda)$.
2. In [Nakai-Nakanishi, 06], they have conjectured some formulas for q-characters of $L_q(\lambda)$ (q-character $\stackrel{\text{specialize}}{\rightarrow}$ character). In fact the specialization of their formula coincides with $\det\left(\mathrm{ch}\, L_q((\lambda_i - i + j)\varpi_1)\right)_{1\le i,j\le n}$.
3. In type B, NN conj. has been proven by [Hernandez, 07].
4. In type CD, any closed character formula for minimal affinizations has not been obtained before (except for some special ones such as KR modules).

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Sketch of the proof

Graded limits

$L_q(\lambda)$: $U_q(\mathcal{L}\mathfrak{g})$-mod./$\mathbb{C}(q)$ $\xrightarrow{q\to 1}$ $L_1(\lambda)$: $\mathcal{L}\mathfrak{g}$-mod./$\mathbb{C}$ (classical limit)

$\xrightarrow{\text{restrict}}$ $L_1(\lambda)$: $\mathfrak{g}[t]$-module $\quad \big(\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t] \subseteq \mathcal{L}\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]\big)$

Fact. $^\exists a \in \mathbb{C}^\times$ s.t. $\big(\mathfrak{g} \otimes (t+a)^N\big) L_1(\lambda) = 0 \quad (N \gg 0)$

$\rightsquigarrow$ Define an auto. τ_a on $\mathfrak{g}[t]$ by $\tau_a\big(g \otimes f(t)\big) = g \otimes f(t+a)$

$L(\lambda) := \tau_a^*\big(L_1(\lambda)\big)$: **graded limit** of $L_q(\lambda)$ ($\mathbb{Z}$-graded $\mathfrak{g}[t]$-module)

Remark. $\mathrm{ch}\, L_q(\lambda) = \mathrm{ch}\, L(\lambda)$.

Sketch of the proof

$\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$: triangular decomosition,

Define $\Delta'_+ := \{\alpha \in \Delta_+ \mid \alpha = \sum m_i\alpha_i,\ m_i \leq 1\} \subseteq \Delta_+$.

Theorem (N)

Let $M(\lambda)$ be the $\mathfrak{g}[t]$-module generated by a vector v with relations

$$\mathfrak{n}_+[t]v = 0, \quad (h \otimes t^n)v = \delta_{0,n}\lambda(h)v \text{ for } h \in \mathfrak{h}, \quad f_i^{\lambda(h_i)+1}v = 0,$$

$$(f_\alpha \otimes t)v = 0 \text{ for } \alpha \in \Delta'_+.$$

Then the graded limit $L(\lambda)$ is isomorphic to $M(\lambda)$.

Sketch of the proof

Theorem (Chari-Greenstein, 11)

$$\sum_{(\lambda,s)\in\Gamma(\mu)} (-1)^s \dim \mathrm{Hom}_{\mathfrak{g}}\Big(V(\lambda), \bigwedge^s \mathfrak{g} \otimes V(\mu)\Big) \mathrm{ch}\, M(\lambda) = \mathrm{ch}\, V(\mu),$$

$\Gamma(\mu) = \{(\lambda, s) \mid \mu = \lambda + \sum_{\alpha\notin\Delta'_+} n_\alpha \alpha,\ \sum n_\alpha = s\} \subseteq P^+ \times \mathbb{Z}_{\geq 0}.$

Theorem (Sam, 14)

Setting $H_\lambda = \det\Big(\mathrm{ch}\, L_q\big((\lambda_i - i + j)\varpi_1\big)\Big)_{1\leq i,j\leq n}$,

$$\sum_{(\lambda,s)\in\Gamma(\mu)} (-1)^s \dim \mathrm{Hom}_{\mathfrak{g}}\Big(V(\lambda), \bigwedge^s \mathfrak{g} \otimes V(\mu)\Big) H_\lambda = \mathrm{ch}\, V(\mu).$$

$\therefore H_\lambda = \mathrm{ch}\, M(\lambda) = \mathrm{ch}\, L(\lambda) = \mathrm{ch}\, L_q(\lambda).$

Sketch of the proof

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