Demazure modules, Demazure crystals and the $X = M$ conjecture

Katsuyuki Naoi

Graduate School of Mathematical Sciences
University of Tokyo

October 11th, 2011

Plan of the talk

1. Relations between Demazure crystals and KR crystals.
   (i) Previous result by Schilling and Tingley.
   (ii) Main result.
2. Application: $X = M$ conjecture for $A_n^{(1)}$ and $D_n^{(1)}$.
   (i) What is the $X = M$ conjecture?
   (ii) The skech of the proof.

Plan of the talk

1. Relations between Demazure crystals and KR crystals.
   (i) Previous result by Schilling and Tingley.
   (ii) Main result.
2. Application: $X = M$ conjecture for $A_n^{(1)}$ and $D_n^{(1)}$.
   (i) What is the $X = M$ conjecture?
   (ii) The skech of the proof.

Plan of the talk

1. Relations between Demazure crystals and KR crystals.
   (i) Previous result by Schilling and Tingley.
   (ii) Main result.
2. Application: $X = M$ conjecture for $A_n^{(1)}$ and $D_n^{(1)}$.
   (i) What is the $X = M$ conjecture?
   (ii) The skech of the proof.

Notation

$\mathfrak{g}$: affine Lie algebra, $I = \{0, \ldots, n\}$, $I_0 = I \setminus \{0\}$,

$\mathfrak{g}_0 \subseteq \mathfrak{g}$: simple Lie subalgebra corresponding to I_0,

W, W_0: Weyl groups, $w_0 \in W_0$: longest element,

P^+, P_0^+: sets of dominant integral weights,

$U_q(\mathfrak{g}), U_q(\mathfrak{g}_0)$: quantized enveloping algebras,

$U_q'(\mathfrak{g}) \subseteq U_q(\mathfrak{g})$: quantum affine algebra without the degree operator,

$\Lambda_i \in P^+$ ($i \in I$): fundamental weights of $\mathfrak{g}$,

$\varpi_i \in P_0^+$ ($i \in I_0$): fundamental weight of $\mathfrak{g}_0$.

$B(\Lambda)$: crystal basis of the integrable highest weight $U_q(\mathfrak{g})$-module with highest weight $\Lambda \in P^+$,

$u_\Lambda \subseteq B(\Lambda)$: highest weight element.

Theorem

If finite $U'_q(\mathfrak{g})$-crystal B is perfect (some technical condition), then we have an isomorphism of $U'_q(\mathfrak{g})$-crystals

$$B(\Lambda) \otimes B \cong B(\Lambda')$$

for suitable $\Lambda, \Lambda' \in P^+$.

Question: What is the **image of $u_\Lambda \otimes B$** under the above isomorphism?

Answer: Demazure crystal (recalled below).

$B(\Lambda)$: crystal basis of the integrable highest weight
$U_q(\mathfrak{g})$-module with highest weight $\Lambda \in P^+$,
$u_\Lambda \subseteq B(\Lambda)$: highest weight element.

Theorem

If finite $U'_q(\mathfrak{g})$-crystal B is perfect (some technical condition), then we have an isomorphism of $U'_q(\mathfrak{g})$-crystals

$$B(\Lambda) \otimes B \cong B(\Lambda')$$

for suitable $\Lambda, \Lambda' \in P^+$.

Question: What is the **image of $u_\Lambda \otimes B$** under the above isomorphism?

Answer: Demazure crystal (recalled below).

$B(\Lambda)$: crystal basis of the integrable highest weight $U_q(\mathfrak{g})$-module with highest weight $\Lambda \in P^+$,

$u_\Lambda \subseteq B(\Lambda)$: highest weight element.

Theorem

If finite $U'_q(\mathfrak{g})$-crystal B is perfect (some technical condition), then we have an isomorphism of $U'_q(\mathfrak{g})$-crystals

$$B(\Lambda) \otimes B \cong B(\Lambda')$$

for suitable $\Lambda, \Lambda' \in P^+$.

Question: What is the **image of $u_\Lambda \otimes B$** under the above isomorphism?

Answer: Demazure crystal (recalled below).

$B(\Lambda)$: crystal basis of the integrable highest weight $U_q(\mathfrak{g})$-module with highest weight $\Lambda \in P^+$,

$u_\Lambda \subseteq B(\Lambda)$: highest weight element.

Theorem

If finite $U'_q(\mathfrak{g})$-crystal B is perfect (some technical condition), then we have an isomorphism of $U'_q(\mathfrak{g})$-crystals

$$B(\Lambda) \otimes B \cong B(\Lambda')$$

for suitable $\Lambda, \Lambda' \in P^+$.

Question: What is the **image of** $u_\Lambda \otimes B$ under the above isomorphism?

Answer: Demazure crystal (recalled below).

$W^{r,\ell}$ $(r \in I_0, \ell \in \mathbb{Z}_{>0})$: Kirillov-Reshetikhin (KR) modules

: a class of irreducible finite-dimensional $U'_q(\mathfrak{g})$-modules.

Theorem ([Okado, Schilling], [Fourier, Okado, Schilling])

(i) *If $\mathfrak{g}$ is nonexceptional, $W^{r,\ell}$ has a crystal basis $B^{r,\ell}$ for each r, ℓ ($B^{r,\ell}$: KR crystal).*

(ii) *For each $r \in I_0$, $c_r \in \{1, 2, 3\}$ exists such that*

$$B^{r,\ell} \text{ is perfect } \iff \ell \in \mathbb{Z}_{>0} c_r.$$

Moreover if $\mathfrak{g}$ is simply-laced or twisted, then all c_r are 1.

$\Longrightarrow$ For any sequence $r_1, \ldots, r_p \in I_0$ and $\ell \in \mathbb{Z}_{>0}$,

$B^{r_1, c_{r_1}\ell} \otimes \cdots \otimes B^{r_p, c_{r_p}\ell}$ is perfect.

Kirillov-Reshetikhin crystal

$W^{r,\ell}$ $(r \in I_0, \ell \in \mathbb{Z}_{>0})$: Kirillov-Reshetikhin (KR) modules

: a class of irreducible finite-dimensional $U_q'(\mathfrak{g})$-modules.

Theorem ([Okado, Schilling], [Fourier, Okado, Schilling])

(i) *If $\mathfrak{g}$ is nonexceptional, $W^{r,\ell}$ has a crystal basis $B^{r,\ell}$ for each r, ℓ ($B^{r,\ell}$: KR crystal).*

(ii) *For each $r \in I_0$, $c_r \in \{1, 2, 3\}$ exists such that*

$$B^{r,\ell} \text{ is perfect } \iff \ell \in \mathbb{Z}_{>0} c_r.$$

Moreover if $\mathfrak{g}$ is simply-laced or twisted, then all c_r are 1.

$\Longrightarrow$ For any sequence $r_1, \ldots, r_p \in I_0$ and $\ell \in \mathbb{Z}_{>0}$,

$B^{r_1, c_{r_1}\ell} \otimes \cdots \otimes B^{r_p, c_{r_p}\ell}$ is perfect.

$W^{r,\ell}$ $(r \in I_0, \ell \in \mathbb{Z}_{>0})$: Kirillov-Reshetikhin (KR) modules

: a class of irreducible finite-dimensional $U_q'(\mathfrak{g})$-modules.

Theorem ([Okado, Schilling], [Fourier, Okado, Schilling])

(i) *If $\mathfrak{g}$ is nonexceptional, $W^{r,\ell}$ has a crystal basis $B^{r,\ell}$ for each r, ℓ ($B^{r,\ell}$: KR crystal).*

(ii) *For each $r \in I_0$, $c_r \in \{1, 2, 3\}$ exists such that*

$$B^{r,\ell} \text{ is perfect } \iff \ell \in \mathbb{Z}_{>0} c_r.$$

Moreover if $\mathfrak{g}$ is simply-laced or twisted, then all c_r are 1.

$\Longrightarrow$ For any sequence $r_1, \ldots, r_p \in I_0$ and $\ell \in \mathbb{Z}_{>0}$, $B^{r_1, c_{r_1}\ell} \otimes \cdots \otimes B^{r_p, c_{r_p}\ell}$ is perfect.

For a crystal B, a subset $S \subseteq B$ and $i \in I$, we denote by $F_i(S)$ the subset

$$F_i(S) = \{\tilde{f}_i^k(b) \mid b \in S, k \geq 0\} \setminus \{0\} \subseteq B.$$

Let $w \in W$ with a reduced expression $w = s_{i_k} \cdots s_{i_1}$. It is known that the subset

$$B_w(\Lambda) = F_{i_k} \cdots F_{i_1}(u_\Lambda) \subseteq B(\Lambda)$$

does not depend on the choice of the expression.

Definition (Kashiwara, '93)

$B_w(\Lambda)$ *is called a Demazure crystal.*

For a crystal B, a subset $S \subseteq B$ and $i \in I$, we denote by $F_i(S)$ the subset

$$F_i(S) = \{\tilde{f}_i^k(b) \mid b \in S, k \geq 0\} \setminus \{0\} \subseteq B.$$

Let $w \in W$ with a reduced expression $w = s_{i_k} \cdots s_{i_1}$. It is known that the subset

$$B_w(\Lambda) = F_{i_k} \cdots F_{i_1}(u_\Lambda) \subseteq B(\Lambda)$$

does not depend on the choice of the expression.

Definition (Kashiwara, '93)

$B_w(\Lambda)$ is called a Demazure crystal.

Demazure crystal

For a crystal B, a subset $S \subseteq B$ and $i \in I$, we denote by $F_i(S)$ the subset

$$F_i(S) = \{\tilde{f}_i^k(b) \mid b \in S, k \geq 0\} \setminus \{0\} \subseteq B.$$

Let $w \in W$ with a reduced expression $w = s_{i_k} \cdots s_{i_1}$. It is known that the subset

$$B_w(\Lambda) = F_{i_k} \cdots F_{i_1}(u_\Lambda) \subseteq B(\Lambda)$$

does not depend on the choice of the expression.

Definition (Kashiwara, '93)

$B_w(\Lambda)$ is called a Demazure crystal.

For a subset S of a crystal, we denote its character by

$$\operatorname{ch} S = \sum_{b \in S} e^{\mathrm{wt}(b)} \in \mathbb{Z}[P].$$

Theorem ([Kashiwara])

$$\operatorname{ch} B_w(\Lambda) = D_w(e^{\Lambda}).$$

If $w = s_{i_k} \cdots s_{i_1}$ is a reduced expression, D_w is defined by $D_w = D_{i_k} \cdots D_{i_1}$ where

$$D_i(e^{\Lambda}) = \begin{cases} e^{s_i(\Lambda)} + \cdots + e^{\Lambda} & \text{if } \langle \Lambda, \alpha_i^{\vee} \rangle \geq 0, \\ 0 & \text{if } \langle \Lambda, \alpha_i^{\vee} \rangle = -1, \\ -e^{\Lambda+\alpha_i} - \cdots - e^{s_i(\Lambda)-\alpha_i} & \text{if } \langle \Lambda, \alpha_i^{\vee} \rangle \leq -2. \end{cases}$$

For a subset S of a crystal, we denote its character by

$$\mathrm{ch}\, S = \sum_{b \in S} e^{\mathrm{wt}(b)} \in \mathbb{Z}[P].$$

Theorem ([Kashiwara])

$$\mathrm{ch}\, B_w(\Lambda) = D_w(e^{\Lambda}).$$

If $w = s_{i_k} \cdots s_{i_1}$ is a reduced expression, D_w is defined by $D_w = D_{i_k} \cdots D_{i_1}$ where

$$D_i(e^{\Lambda}) = \begin{cases} e^{s_i(\Lambda)} + \cdots + e^{\Lambda} & \text{if } \langle \Lambda, \alpha_i^{\vee} \rangle \geq 0, \\ 0 & \text{if } \langle \Lambda, \alpha_i^{\vee} \rangle = -1, \\ -e^{\Lambda+\alpha_i} - \cdots - e^{s_i(\Lambda)-\alpha_i} & \text{if } \langle \Lambda, \alpha_i^{\vee} \rangle \leq -2. \end{cases}$$

For a subset S of a crystal, we denote its character by

$$\operatorname{ch} S = \sum_{b \in S} e^{\mathrm{wt}(b)} \in \mathbb{Z}[P].$$

Theorem ([Kashiwara])

$$\operatorname{ch} B_w(\Lambda) = D_w(e^{\Lambda}).$$

If $w = s_{i_k} \cdots s_{i_1}$ is a reduced expression, D_w is defined by $D_w = D_{i_k} \cdots D_{i_1}$ where

$$D_i(e^{\Lambda}) = \begin{cases} e^{s_i(\Lambda)} + \cdots + e^{\Lambda} & \text{if } \langle \Lambda, \alpha_i^{\vee} \rangle \geq 0, \\ 0 & \text{if } \langle \Lambda, \alpha_i^{\vee} \rangle = -1, \\ -e^{\Lambda+\alpha_i} - \cdots - e^{s_i(\Lambda)-\alpha_i} & \text{if } \langle \Lambda, \alpha_i^{\vee} \rangle \leq -2. \end{cases}$$

Previous result

Assume that $\mathfrak{g}$ is nonexceptional. For given $r_1, \dots, r_p \in I_0$ and $\ell \in \mathbb{Z}_{>0}$, set

$$B = B^{r_1, c_{r_1}\ell} \otimes \cdots \otimes B^{r_p, c_{r_p}\ell},$$

and let $i \in I$ and $w \in W$ be elements satisfying

$$w\Lambda_i = w_0(c_{r_1}\varpi_{r_1} + \cdots + c_{r_p}\varpi_{r_p}) + \Lambda_0.$$

Then we have $B(\ell\Lambda_0) \otimes B \xrightarrow{\sim} B(\ell\Lambda_i)$ as $U_q'(\mathfrak{g})$-crystals.

Theorem (Schilling and Tingley, 2011)

(1) *The image of $u_{\ell\Lambda_0} \otimes B$ under the above isomorphism is $B_w(\ell\Lambda_i)$.*

(2) *The weight of the image of $u_{\ell\Lambda_0} \otimes b$ is equal to $\mathrm{wt}(b) - \delta D(b)$, where $D : B \to \mathbb{Z}$ is the energy function.*

Previous result

Assume that $\mathfrak{g}$ is nonexceptional. For given $r_1, \ldots, r_p \in I_0$ and $\ell \in \mathbb{Z}_{>0}$, set

$$B = B^{r_1, c_{r_1}\ell} \otimes \cdots \otimes B^{r_p, c_{r_p}\ell},$$

and let $i \in I$ and $w \in W$ be elements satisfying

$$w\Lambda_i = w_0(c_{r_1}\varpi_{r_1} + \cdots + c_{r_p}\varpi_{r_p}) + \Lambda_0.$$

Then we have $B(\ell\Lambda_0) \otimes B \xrightarrow{\sim} B(\ell\Lambda_i)$ as $U_q'(\mathfrak{g})$-crystals.

Theorem (Schilling and Tingley, 2011)

(1) *The image of* $u_{\ell\Lambda_0} \otimes B$ *under the above isomorphism is* $B_w(\ell\Lambda_i)$.

(2) *The weight of the image of* $u_{\ell\Lambda_0} \otimes b$ *is equal to* $\mathrm{wt}(b) - \delta D(b)$, *where* $D : B \to \mathbb{Z}$ *is the energy function.*

Previous result

Assume that $\mathfrak{g}$ is nonexceptional. For given $r_1, \ldots, r_p \in I_0$ and $\ell \in \mathbb{Z}_{>0}$, set

$$B = B^{r_1, c_{r_1}\ell} \otimes \cdots \otimes B^{r_p, c_{r_p}\ell},$$

and let $i \in I$ and $w \in W$ be elements satisfying

$$w\Lambda_i = w_0(c_{r_1}\varpi_{r_1} + \cdots + c_{r_p}\varpi_{r_p}) + \Lambda_0.$$

Then we have $B(\ell\Lambda_0) \otimes B \xrightarrow{\sim} B(\ell\Lambda_i)$ as $U'_q(\mathfrak{g})$-crystals.

Theorem (Schilling and Tingley, 2011)

(1) *The image of $u_{\ell\Lambda_0} \otimes B$ under the above isomorphism is $B_w(\ell\Lambda_i)$.*

(2) *The weight of the image of $u_{\ell\Lambda_0} \otimes b$ is equal to $\mathrm{wt}(b) - \delta D(b)$, where $D : B \to \mathbb{Z}$ is the energy function.*

The precise meaning of the second statement

Let $\Psi : u_{\ell\Lambda_0} \otimes B \xrightarrow{\sim} B_w(\ell\Lambda_i)$ be the isomorphism. Since $B(\ell\Lambda_i)$ is a $U_q(\mathfrak{g})$-crystal, for each element $b \in B$ we have

$$\mathrm{wt}(\Psi(u_{\ell\Lambda_0}\otimes b)) = \lambda+\ell\Lambda_0+s\delta \in P \quad \text{for some } \lambda \in P_0 \text{ and } s \in \mathbb{Z}$$

(δ is the null root).

On the other hand since $B(\ell\Lambda_0) \otimes B$ is a $U'_q(\mathfrak{g})$-crystal, we have

$$\mathrm{wt}(u_{\ell\Lambda_0} \otimes b) = \lambda + \ell\Lambda_0 \in P/\mathbb{Z}\delta.$$

The second statment says that a function $D : B \to \mathbb{Z}$ called energy function is defined, and it satisfies that

$$D(b) = -s.$$

The precise meaning of the second statement

Let $\Psi : u_{\ell\Lambda_0} \otimes B \xrightarrow{\sim} B_w(\ell\Lambda_i)$ be the isomorphism. Since $B(\ell\Lambda_i)$ is a $U_q(\mathfrak{g})$-crystal, for each element $b \in B$ we have

$$\mathrm{wt}(\Psi(u_{\ell\Lambda_0} \otimes b)) = \lambda + \ell\Lambda_0 + s\delta \in P \quad \text{for some } \lambda \in P_0 \text{ and } s \in \mathbb{Z}$$

(δ is the null root).

On the other hand since $B(\ell\Lambda_0) \otimes B$ is a $U'_q(\mathfrak{g})$-crystal, we have

$$\mathrm{wt}(u_{\ell\Lambda_0} \otimes b) = \lambda + \ell\Lambda_0 \in P/\mathbb{Z}\delta.$$

The second statment says that a function $D : B \to \mathbb{Z}$ called energy function is defined, and it satisfies that

$$D(b) = -s.$$

The precise meaning of the second statement

Let $\Psi : u_{\ell\Lambda_0} \otimes B \xrightarrow{\sim} B_w(\ell\Lambda_i)$ be the isomorphism. Since $B(\ell\Lambda_i)$ is a $U_q(\mathfrak{g})$-crystal, for each element $b \in B$ we have

$$\mathrm{wt}(\Psi(u_{\ell\Lambda_0} \otimes b)) = \lambda + \ell\Lambda_0 + s\delta \in P \quad \text{for some } \lambda \in P_0 \text{ and } s \in \mathbb{Z}$$

(δ is the null root).

On the other hand since $B(\ell\Lambda_0) \otimes B$ is a $U'_q(\mathfrak{g})$-crystal, we have

$$\mathrm{wt}(u_{\ell\Lambda_0} \otimes b) = \lambda + \ell\Lambda_0 \in P/\mathbb{Z}\delta.$$

The second statment says that a function $D : B \to \mathbb{Z}$ called energy function is defined, and it satisfies that

$$D(b) = -s.$$

The precise meaning of the second statement

Let $\Psi : u_{\ell\Lambda_0} \otimes B \xrightarrow{\sim} B_w(\ell\Lambda_i)$ be the isomorphism. Since $B(\ell\Lambda_i)$ is a $U_q(\mathfrak{g})$-crystal, for each element $b \in B$ we have

$$\mathrm{wt}(\Psi(u_{\ell\Lambda_0}\otimes b)) = \lambda+\ell\Lambda_0+s\delta \in P \quad \text{for some } \lambda \in P_0 \text{ and } s \in \mathbb{Z}$$

(δ is the null root).

On the other hand since $B(\ell\Lambda_0) \otimes B$ is a $U'_q(\mathfrak{g})$-crystal, we have

$$\mathrm{wt}(u_{\ell\Lambda_0} \otimes b) = \lambda + \ell\Lambda_0 \in P/\mathbb{Z}\delta.$$

The second statment says that a function $D : B \to \mathbb{Z}$ called energy function is defined, and it satisfies that

$$D(b) = -s.$$

Definition of the energy function

Proposition (combinatorial R-matrix)

For every KR crystals B_1, B_2, $^{!\exists}R : B_1 \otimes B_2 \xrightarrow{\sim} B_2 \otimes B_1$.

$H : B_1 \otimes B_2 \to \mathbb{Z}$ (local energy function)

$\overset{\text{def}}{\Longleftrightarrow}$ ∘ Constant on each $U_q(\mathfrak{g}_0)$-component,

∘ For $b_1 \otimes b_2 \in B_1 \otimes B_2$, $R(b_1 \otimes b_2) = \tilde{b}_2 \otimes \tilde{b}_1$,

$$H(e_0(b_1 \otimes b_2)) = \begin{cases} H(b_1 \otimes b_2) + 1 & e_0(b_1 \otimes b_2) = e_0 b_1 \otimes b_2, \\ & e_0(\tilde{b}_2 \otimes \tilde{b}_1) = e_0 \tilde{b}_2 \otimes \tilde{b}_1, \\ H(b_1 \otimes b_2) - 1 & e_0(b_1 \otimes b_2) = b_1 \otimes e_0 b_2, \\ & e_0(\tilde{b}_2 \otimes \tilde{b}_1) = \tilde{b}_2 \otimes e_0 \tilde{b}_1, \\ H(b_1 \otimes b_2) & \text{otherwise.} \end{cases}$$

Proposition (combinatorial R-matrix)

For every KR crystals B_1, B_2, $^{!\exists}R : \ B_1 \otimes B_2 \xrightarrow{\sim} B_2 \otimes B_1$.

$H : B_1 \otimes B_2 \to \mathbb{Z}$ (local energy function)

$\overset{\text{def}}{\Longleftrightarrow}$ ∘ Constant on each $U_q(\mathfrak{g}_0)$-component,

∘ For $b_1 \otimes b_2 \in B_1 \otimes B_2$, $R(b_1 \otimes b_2) = \tilde{b}_2 \otimes \tilde{b}_1$,

$$H(e_0(b_1 \otimes b_2)) = \begin{cases} H(b_1 \otimes b_2) + 1 & e_0(b_1 \otimes b_2) = e_0 b_1 \otimes b_2, \\ & e_0(\tilde{b}_2 \otimes \tilde{b}_1) = e_0 \tilde{b}_2 \otimes \tilde{b}_1, \\ H(b_1 \otimes b_2) - 1 & e_0(b_1 \otimes b_2) = b_1 \otimes e_0 b_2, \\ & e_0(\tilde{b}_2 \otimes \tilde{b}_1) = \tilde{b}_2 \otimes e_0 \tilde{b}_1, \\ H(b_1 \otimes b_2) & \text{otherwise.} \end{cases}$$

Proposition (combinatorial R-matrix)

For every KR crystals B_1, B_2, $^{!\exists}R : \; B_1 \otimes B_2 \xrightarrow{\sim} B_2 \otimes B_1$.

$H : B_1 \otimes B_2 \to \mathbb{Z}$ (local energy function)

$\overset{\text{def}}{\iff}$ ∘ Constant on each $U_q(\mathfrak{g}_0)$-component,

∘ For $b_1 \otimes b_2 \in B_1 \otimes B_2$, $R(b_1 \otimes b_2) = \tilde{b}_2 \otimes \tilde{b}_1$,

$$H(e_0(b_1 \otimes b_2)) = \begin{cases} H(b_1 \otimes b_2) + 1 & e_0(b_1 \otimes b_2) = e_0 b_1 \otimes b_2, \\ & e_0(\tilde{b}_2 \otimes \tilde{b}_1) = e_0 \tilde{b}_2 \otimes \tilde{b}_1, \\ H(b_1 \otimes b_2) - 1 & e_0(b_1 \otimes b_2) = b_1 \otimes e_0 b_2, \\ & e_0(\tilde{b}_2 \otimes \tilde{b}_1) = \tilde{b}_2 \otimes e_0 \tilde{b}_1, \\ H(b_1 \otimes b_2) & \text{otherwise.} \end{cases}$$

$D : B \to \mathbb{Z}$ (energy function)

$\overset{\text{def}}{\Longleftrightarrow}$ (1) In the case where $B = B^{r,s}$:

$$D(b) := H(b^\natural \otimes b) \quad \text{for some special element } b^\natural \in B.$$

(2) In the case where $B = B_1 \otimes \cdots \otimes B_p$:

For $b_1 \otimes \cdots \otimes b_p \in B$ and $1 \le i \le j \le p$, define $b_j^{(i)} \in B_j$ by

$$\begin{aligned} B_i \otimes B_{i+1} \otimes \cdots \otimes B_j &\xrightarrow{\sim} B_j \otimes B_i \otimes \cdots \otimes B_{j-1} \\ b_i \otimes b_{i+1} \otimes \cdots \otimes b_j &\mapsto b_j^{(i)} \otimes \tilde{b}_i \otimes \cdots \otimes \tilde{b}_{j-1}. \end{aligned}$$

Then $D : B \to \mathbb{Z}$ is defined by

$$D(b_1 \otimes \cdots \otimes b_p) := \sum_{1 \le i \le p} D(b_i^{(1)}) + \sum_{1 \le i < j \le p} H(b_i \otimes b_j^{(i+1)}).$$

$D : B \to \mathbb{Z}$ (energy function)

$\overset{\text{def}}{\Longleftrightarrow}$ (1) In the case where $B = B^{r,s}$:

$$D(b) := H(b^\natural \otimes b) \quad \text{for some special element } b^\natural \in B.$$

(2) In the case where $B = B_1 \otimes \cdots \otimes B_p$:

For $b_1 \otimes \cdots \otimes b_p \in B$ and $1 \le i \le j \le p$, define $b_j^{(i)} \in B_j$ by

$$\begin{aligned} B_i \otimes B_{i+1} \otimes \cdots \otimes B_j &\xrightarrow{\sim} B_j \otimes B_i \otimes \cdots \otimes B_{j-1} \\ b_i \otimes b_{i+1} \otimes \cdots \otimes b_j &\mapsto b_j^{(i)} \otimes \tilde{b}_i \otimes \cdots \otimes \tilde{b}_{j-1}. \end{aligned}$$

Then $D : B \to \mathbb{Z}$ is defined by

$$D(b_1 \otimes \cdots \otimes b_p) := \sum_{1 \le i \le p} D(b_i^{(1)}) + \sum_{1 \le i < j \le p} H(b_i \otimes b_j^{(i+1)}).$$

$D : B \to \mathbb{Z}$ (energy function)

$\overset{\text{def}}{\iff}$ (1) In the case where $B = B^{r,s}$:

$$D(b) := H(b^\natural \otimes b) \quad \text{for some special element } b^\natural \in B.$$

(2) In the case where $B = B_1 \otimes \cdots \otimes B_p$:

For $b_1 \otimes \cdots \otimes b_p \in B$ and $1 \le i \le j \le p$, define $b_j^{(i)} \in B_j$ by

$$\begin{aligned} B_i \otimes B_{i+1} \otimes \cdots \otimes B_j &\xrightarrow{\sim} B_j \otimes B_i \otimes \cdots \otimes B_{j-1} \\ b_i \otimes b_{i+1} \otimes \cdots \otimes b_j &\mapsto b_j^{(i)} \otimes \tilde{b}_i \otimes \cdots \otimes \tilde{b}_{j-1}. \end{aligned}$$

Then $D : B \to \mathbb{Z}$ is defined by

$$D(b_1 \otimes \cdots \otimes b_p) := \sum_{1 \le i \le p} D(b_i^{(1)}) + \sum_{1 \le i < j \le p} H(b_i \otimes b_j^{(i+1)}).$$

$D : B \to \mathbb{Z}$ (energy function)

$\overset{\text{def}}{\Longleftrightarrow}$ (1) In the case where $B = B^{r,s}$:

$$D(b) := H(b^\natural \otimes b) \quad \text{for some special element } b^\natural \in B.$$

(2) In the case where $B = B_1 \otimes \cdots \otimes B_p$:

For $b_1 \otimes \cdots \otimes b_p \in B$ and $1 \leq i \leq j \leq p$, define $b_j^{(i)} \in B_j$ by

$$\begin{aligned} B_i \otimes B_{i+1} \otimes \cdots \otimes B_j &\xrightarrow{\sim} B_j \otimes B_i \otimes \cdots \otimes B_{j-1} \\ b_i \otimes b_{i+1} \otimes \cdots \otimes b_j &\mapsto b_j^{(i)} \otimes \tilde{b}_i \otimes \cdots \otimes \tilde{b}_{j-1}. \end{aligned}$$

Then $D : B \to \mathbb{Z}$ is defined by

$$D(b_1 \otimes \cdots \otimes b_p) := \sum_{1 \leq i \leq p} D(b_i^{(1)}) + \sum_{1 \leq i < j \leq p} H(b_i \otimes b_j^{(i+1)}).$$

Rephrase the above theorem

Theorem

Set $B = B^{r_1, c_{r_1}\ell} \otimes \cdots \otimes B^{r_p, c_{r_p}\ell}$, and let $i \in I$ and $w \in W$ be elements such that

$$w(\Lambda_i) = w_0(c_{r_1}\varpi_{r_1} + \cdots + c_{r_p}\varpi_{r_p}) + \Lambda_0.$$

Then there exists an isomorphism of full subgraphs

$$\Psi : u_{\ell\Lambda_0} \otimes B \xrightarrow{\sim} B_w(\ell\Lambda_i)$$

which satisfies

$$\mathrm{wt}\,\Psi(u_{\ell\Lambda_0} \otimes b) = \mathrm{wt}(b) - \delta D(b) \quad \text{for } b \in B.$$

Rephrase the above theorem

Theorem

Set $B = B^{r_1, c_{r_1}\ell} \otimes \cdots \otimes B^{r_p, c_{r_p}\ell}$*, and let* $i \in I$ *and* $w \in W$ *be elements such that*

$$w(\Lambda_i) = w_0(c_{r_1}\varpi_{r_1} + \cdots + c_{r_p}\varpi_{r_p}) + \Lambda_0.$$

Then there exists an isomorphism of full subgraphs

$$\Psi : u_{\ell\Lambda_0} \otimes B \tilde{\rightarrow} B_w(\ell\Lambda_i)$$

which satisfies

$$\mathrm{wt}\,\Psi(u_{\ell\Lambda_0} \otimes b) = \mathrm{wt}(b) - \delta D(b) \quad \text{for } b \in B.$$

As a consequence of the above theorem, we obtain the following corollary:

Corollary

$$\sum_{b \in B} e^{\mathrm{wt}(b) - \delta D(b)} = \mathrm{ch}\, B_w(\ell\Lambda_i) = D_w(e^{\ell\Lambda_i}).$$

Goal: Generalize the above results to

$$B = B^{r_1, c_{r_1}\ell_1} \otimes \cdots \otimes B^{r_p, c_{r_p}\ell_p}$$

for arbitrary $\ell_1, \ldots, \ell_p \in \mathbb{Z}_{>0}$.

Since it is **not perfect**,

$$B(\Lambda) \otimes B \not\cong B(\Lambda')$$

for any $\Lambda, \Lambda' \in P^+$.

As a consequence of the above theorem, we obtain the following corollary:

Corollary

$$\sum_{b \in B} e^{\mathrm{wt}(b) - \delta D(b)} = \mathrm{ch}\, B_w(\ell \Lambda_i)$$
$$= D_w(e^{\ell \Lambda_i}).$$

Goal: Generalize the above results to

$$B = B^{r_1, c_{r_1} \ell_1} \otimes \cdots \otimes B^{r_p, c_{r_p} \ell_p}$$

for arbitrary $\ell_1, \ldots, \ell_p \in \mathbb{Z}_{>0}$.

Since it is **not perfect**,

$$B(\Lambda) \otimes B \ncong B(\Lambda')$$

for any $\Lambda, \Lambda' \in P^+$.

As a consequence of the above theorem, we obtain the following corollary:

Corollary

$$\sum_{b \in B} e^{\mathrm{wt}(b) - \delta D(b)} = \mathrm{ch}\, B_w(\ell \Lambda_i)$$

$$= D_w(e^{\ell \Lambda_i}).$$

Goal: Generalize the above results to

$$B = B^{r_1, c_{r_1} \ell_1} \otimes \cdots \otimes B^{r_p, c_{r_p} \ell_p}$$

for arbitrary $\ell_1, \ldots, \ell_p \in \mathbb{Z}_{>0}$.

Since it is **not perfect**,

$$B(\Lambda) \otimes B \not\cong B(\Lambda')$$

for any $\Lambda, \Lambda' \in P^+$.

Main theorem: a generalization of the above result

Assume that $\mathfrak{g}$ is nonexceptional. For simplicity, we also assume that the tensor product

$$B = B^{r_1, c_{r_1}\ell_1} \otimes \cdots \otimes B^{r_p, c_{r_p}\ell_p}$$

satisfies $\ell_1 \geq \cdots \geq \ell_p$. Define $i_1, \dots, i_p \in I$ and $w_1, \dots, w_p \in W$ by the elements satisfying

$$\begin{aligned} w_1(\Lambda_{i_1}) &= c_{r_1} w_0(\varpi_{r_1}) + \Lambda_0, \\ w_1 w_2(\Lambda_{i_2}) &= w_0(c_{r_1}\varpi_{r_1} + c_{r_2}\varpi_{r_2}) + \Lambda_0, \\ &\vdots \\ w_1 w_2 \cdots w_p(\Lambda_{i_p}) &= w_0(c_{r_1}\varpi_{r_1} + \cdots + c_{r_p}\varpi_{r_p}) + \Lambda_0. \end{aligned}$$

Main theorem: a generalization of the above result

Assume that $\mathfrak{g}$ is nonexceptional. For simplicity, we also assume that the tensor product

$$B = B^{r_1, c_{r_1}\ell_1} \otimes \cdots \otimes B^{r_p, c_{r_p}\ell_p}$$

satisfies $\ell_1 \geq \cdots \geq \ell_p$. Define $i_1, \ldots, i_p \in I$ and $w_1, \ldots, w_p \in W$ by the elements satisfying

$$\begin{aligned} w_1(\Lambda_{i_1}) &= c_{r_1} w_0(\varpi_{r_1}) + \Lambda_0, \\ w_1 w_2(\Lambda_{i_2}) &= w_0(c_{r_1}\varpi_{r_1} + c_{r_2}\varpi_{r_2}) + \Lambda_0, \\ &\vdots \\ w_1 w_2 \cdots w_p(\Lambda_{i_p}) &= w_0(c_{r_1}\varpi_{r_1} + \cdots + c_{r_p}\varpi_{r_p}) + \Lambda_0. \end{aligned}$$

Define a subset

$$S \subseteq B((\ell_1 - \ell_2)\Lambda_{i_1}) \otimes B((\ell_2 - \ell_3)\Lambda_{i_2}) \otimes \cdots \otimes B(\ell_p \Lambda_{i_p})$$

by

$$S = F_{w_1}\Big(u_{(\ell_1-\ell_2)\Lambda_{i_1}} \otimes F_{w_2}(u_{(\ell_2-\ell_3)\Lambda_{i_2}} \otimes \cdots \otimes F_{w_p}(u_{\ell_p\Lambda_{i_p}}) \cdots)\Big).$$

Theorem (N)

There exists an isomorphism of full subgraphs

$$\Psi : u_{\ell_1\Lambda_0} \otimes B^{r_1, c_{r_1}\ell_1} \otimes \cdots \otimes B^{r_p, c_{r_p}\ell_p} \xrightarrow{\sim} S$$

which satisfies

$$\mathrm{wt}\, \Psi(u_{\ell_1\Lambda_0} \otimes b) = \mathrm{wt}(b) - \delta D(b)\delta \quad \textit{for } b \in B.$$

Define a subset

$$S \subseteq B((\ell_1 - \ell_2)\Lambda_{i_1}) \otimes B((\ell_2 - \ell_3)\Lambda_{i_2}) \otimes \cdots \otimes B(\ell_p \Lambda_{i_p})$$

by

$$S = F_{w_1}\Big(u_{(\ell_1-\ell_2)\Lambda_{i_1}} \otimes F_{w_2}(u_{(\ell_2-\ell_3)\Lambda_{i_2}} \otimes \cdots \otimes F_{w_p}(u_{\ell_p\Lambda_{i_p}}) \cdots)\Big).$$

Theorem (N)

There exists an isomorphism of full subgraphs

$$\Psi : u_{\ell_1\Lambda_0} \otimes B^{r_1, c_{r_1}\ell_1} \otimes \cdots \otimes B^{r_p, c_{r_p}\ell_p} \xrightarrow{\sim} S$$

which satisfies

$$\mathrm{wt}\,\Psi(u_{\ell_1\Lambda_0} \otimes b) = \mathrm{wt}(b) - \delta D(b)\delta \quad \textit{for } b \in B.$$

Similarly as a Demazure crystal, the character of S is calculated as follows:

Lemma

$$\mathrm{ch}\, S = D_{w_1}\Big(e^{(\ell_1-\ell_2)\Lambda_{i_1}} \cdot D_{w_2}(e^{(\ell_2-\ell_3)\Lambda_{i_2}} \cdots D_{w_p}(e^{\ell_p\Lambda_{i_p}})\cdots)\Big).$$

Hence we have the following corollary:

Corollary

$$\sum_{b\in B} e^{\mathrm{wt}(b)-\delta D(b)} = \mathrm{ch}\, S$$
$$= D_{w_1}\Big(e^{(\ell_1-\ell_2)\Lambda_{i_1}} \cdot D_{w_2}(e^{(\ell_2-\ell_3)\Lambda_{i_2}} \cdots D_{w_p}(e^{\ell_p\Lambda_{i_p}})\cdots)\Big).$$

Similarly as a Demazure crystal, the character of S is calculated as follows:

Lemma

$$\mathrm{ch}\, S = D_{w_1}\Big(e^{(\ell_1-\ell_2)\Lambda_{i_1}} \cdot D_{w_2}(e^{(\ell_2-\ell_3)\Lambda_{i_2}} \cdots D_{w_p}(e^{\ell_p\Lambda_{i_p}}) \cdots)\Big).$$

Hence we have the following corollary:

Corollary

$$\sum_{b\in B} e^{\mathrm{wt}(b)-\delta D(b)} = \mathrm{ch}\, S$$
$$= D_{w_1}\Big(e^{(\ell_1-\ell_2)\Lambda_{i_1}} \cdot D_{w_2}(e^{(\ell_2-\ell_3)\Lambda_{i_2}} \cdots D_{w_p}(e^{\ell_p\Lambda_{i_p}}) \cdots)\Big).$$

$X = M$ conjecture

For a tensor product of (not necessarily perfect) KR crystals $B = B^{r_1,\ell_1} \otimes \cdots B^{r_p,\ell_p}$ and $\mu \in P_0^+$, we define

$$X(B,\mu,q) = \sum_{b \in B_\mu^{\mathrm{hw}}} q^{D(b)} \qquad \text{(1-dimensional sum)},$$

where B_μ^{hw} is a subset of B defined by

$$B_\mu^{\mathrm{hw}} = \{b \in B \mid \tilde{e}_i(b) = 0 \text{ for } i \in I_0, \mathrm{wt}(b) = \mu\}.$$

Conjecture (Hatayama, Kuniba, et al. '99)

For every $\mu \in P_0^+$, we have

$$X(B,\mu,q) = M(B,\mu,q),$$

where $M(B,\mu,q) \in \mathbb{Z}[q]$ is the fermionic form defined below.

For a tensor product of (not necessarily perfect) KR crystals $B = B^{r_1,\ell_1} \otimes \cdots B^{r_p,\ell_p}$ and $\mu \in P_0^+$, we define

$$X(B, \mu, q) = \sum_{b \in B_\mu^{\mathrm{hw}}} q^{D(b)} \qquad (\text{1-dimensional sum}),$$

where B_μ^{hw} is a subset of B defined by

$$B_\mu^{\mathrm{hw}} = \{b \in B \mid \tilde{e}_i(b) = 0 \text{ for } i \in I_0, \mathrm{wt}(b) = \mu\}.$$

Conjecture (Hatayama, Kuniba, et al. ’99)

For every $\mu \in P_0^+$, we have

$$X(B, \mu, q) = M(B, \mu, q),$$

where $M(B, \mu, q) \in \mathbb{Z}[q]$ is the fermionic form defined below.

For simplicity, assume that $\mathfrak{g}$ is of type $A_n^{(1)}$, $D_n^{(1)}$ or $E_n^{(1)}$.

The fermionic form $M(B,\mu,q)$ is defined as follows:

$$M(B,\mu,q) = \sum_{\substack{m=\{m_u^{(i)}\in\mathbb{Z}_{\geq 0}\}_{i\in I_0,u\geq 1} \\ \text{s.t } p_u^{(i)}\geq 0\ (^\forall i,u), \\ \sum_{i,u} u m_i^{(u)}\alpha_i=\sum_j \ell_j \varpi_{r_j}-\mu}} q^{c(m)} \prod_{i\in I_0,u\geq 1} \begin{bmatrix} p_u^{(i)}+m_u^{(i)} \\ m_u^{(i)} \end{bmatrix}_q ,$$

where

$$c(m) = \frac{1}{2}\sum_{\substack{i,j\in I_0 \\ u,v\geq 1}} (\alpha_i,\alpha_j)\min\{u,v\} m_u^{(i)} m_v^{(j)} - \sum_{u\in\mathbb{Z}_{>0}} \min\{\ell_j,u\} m_u^{(r_j)},$$

$$p_u^{(i)} = \sum_{j\in I_0; r_j=i} \min\{u,\ell_j\} - \sum_{\substack{j\in I_0 \\ v\geq 1}} (\alpha_i,\alpha_j)\min\{u,v\} m_j^{(v)}.$$

($p_u^{(i)}$ is called the vacancy number).

Theorem

The $X = M$ conjecture has been proved in these cases:

- $\mathfrak{g} = A_n^{(1)}$, [Kirillov, Schilling, Shimozono, 2002],
- $\mathfrak{g}$: nonexceptional type, the rank of $\mathfrak{g}$ is sufficiently large, [Lecouvey, Okado, Shimozono, 2010] and [Okado, Sakamoto, 2010],
- $^\forall\mathfrak{g}$, if $\ell_i = 1$ for all i [N],
- Other special cases.

Using the results stated above, we can show the $X = M$ conjecture for type $A_n^{(1)}$ and $D_n^{(1)}$.

Theorem

The $X = M$ conjecture has been proved in these cases:

- $\mathfrak{g} = A_n^{(1)}$, [Kirillov, Schilling, Shimozono, 2002],
- $\mathfrak{g}$: nonexceptional type, the rank of $\mathfrak{g}$ is sufficiently large, [Lecouvey, Okado, Shimozono, 2010] and [Okado, Sakamoto, 2010],
- ${}^{\forall}\mathfrak{g}$, if $\ell_i = 1$ for all i [N],
- Other special cases.

Using the results stated above, we can show the $X = M$ conjecture for type $A_n^{(1)}$ and $D_n^{(1)}$.

Theorem

The $X = M$ conjecture has been proved in these cases:

- $\mathfrak{g} = A_n^{(1)}$, [Kirillov, Schilling, Shimozono, 2002],
- $\mathfrak{g}$: nonexceptional type, the rank of $\mathfrak{g}$ is sufficiently large, [Lecouvey, Okado, Shimozono, 2010] and [Okado, Sakamoto, 2010],
- $^{\forall}\mathfrak{g}$, if $\ell_i = 1$ for all i [N],
- Other special cases.

Using the results stated above, we can show the $X = M$ conjecture for type $A_n^{(1)}$ and $D_n^{(1)}$.

Theorem

The $X = M$ conjecture has been proved in these cases:

- $\mathfrak{g} = A_n^{(1)}$, [Kirillov, Schilling, Shimozono, 2002],
- $\mathfrak{g}$: nonexceptional type, the rank of $\mathfrak{g}$ is sufficiently large, [Lecouvey, Okado, Shimozono, 2010] and [Okado, Sakamoto, 2010],
- $^\forall\mathfrak{g}$, if $\ell_i = 1$ for all i [N],
- Other special cases.

Using the results stated above, we can show the $X = M$ conjecture for type $A_n^{(1)}$ and $D_n^{(1)}$.

Theorem

The $X = M$ conjecture has been proved in these cases:

- $\mathfrak{g} = A_n^{(1)}$, [Kirillov, Schilling, Shimozono, 2002],
- $\mathfrak{g}$: nonexceptional type, the rank of $\mathfrak{g}$ is sufficiently large, [Lecouvey, Okado, Shimozono, 2010] and [Okado, Sakamoto, 2010],
- ${}^{\forall}\mathfrak{g}$, if $\ell_i = 1$ for all i [N],
- Other special cases.

Using the results stated above, we can show the $X = M$ conjecture for type $A_n^{(1)}$ and $D_n^{(1)}$.

For $\mu \in P_0^+$, let $V_0(\mu)$ denote the irreducible $\mathfrak{g}_0$-module.
In order to prove

$$X(B, \mu, q) = M(B, \mu, q)$$

for every $\mu \in P_0^+$, it suffices to show that

$$\sum_{\mu \in P_0^+} X(B, \mu, q) \mathrm{ch}\, V_0(\mu) = \sum_{\mu \in P_0^+} M(B, \mu, q) \mathrm{ch}\, V_0(\mu)$$

since $\mathrm{ch}\, V_0(\mu)$ are linearly independent.

By definition, we have

$$\sum_{\mu \in P_0^+} X(B, \mu, q) \mathrm{ch}\, V_0(\mu) = \sum_{b \in B} q^{D(b)} e^{\mathrm{wt}(b)}.$$

Hence if $\mathfrak{g}$ is nonexceptional, we have from the above corollary that

$$\sum_{\mu \in P_0^+} X(B, \mu, q) \mathrm{ch}\, V_0(\mu)$$
$$= D_{w_1}\Big(e^{(\ell_1 - \ell_2)\Lambda_{i_1}} \cdot D_{w_2}(e^{(\ell_2 - \ell_3)\Lambda_{i_2}} \cdots D_{w_p}(e^{\ell_p \Lambda_{i_p}}) \cdots)\Big),$$

where we set $q = e^{-\delta}$.

By definition, we have

$$\sum_{\mu \in P_0^+} X(B,\mu,q)\mathrm{ch}\, V_0(\mu) = \sum_{b\in B} q^{D(b)} e^{\mathrm{wt}(b)}.$$

Hence if $\mathfrak{g}$ is nonexceptional, we have from the above corollary that

$$\sum_{\mu \in P_0^+} X(B,\mu,q)\mathrm{ch}\, V_0(\mu)$$
$$= D_{w_1}\Big(e^{(\ell_1-\ell_2)\Lambda_{i_1}} \cdot D_{w_2}(e^{(\ell_2-\ell_3)\Lambda_{i_2}} \cdots D_{w_p}(e^{\ell_p \Lambda_{i_p}}) \cdots)\Big),$$

where we set $q = e^{-\delta}$.

On the other hand, the following theorem can be proved:

Theorem (N)

If $\mathfrak{g}$ is of type $A_n^{(1)}$, $D_n^{(1)}$ or $E_n^{(1)}$, then we have

$$\sum_{\mu \in P_0^+} M(B, \mu, q) \mathrm{ch}\, V_0(\mu) = D_{w_1}\Big(e^{(\ell_1 - \ell_2)\Lambda_{i_1}} \cdot D_{w_2}(e^{(\ell_2 - \ell_3)\Lambda_{i_2}} \cdots D_{w_p}(e^{\ell_p \Lambda_{i_p}}) \cdots)\Big),$$

where we set $q = e^{-\delta}$.

sketch of the proof.) Let $V(\Lambda)$ denote the irreducible highest weight $U_q(\mathfrak{g})$-module. We define $\mathcal{S}$ by the subspace of

$$V\big((\ell_1-\ell_2)\Lambda_{i_1}\big)\otimes V\big((\ell_2-\ell_3)\Lambda_{i_2}\big)\otimes\cdots\otimes V(\ell_p\Lambda_{i_p})$$

corresponding to the subset

$$S = F_{w_1}\Big(u_{(\ell_1-\ell_2)\Lambda_{i_1}}\otimes F_{w_2}\big(u_{(\ell_2-\ell_3)\Lambda_{i_2}}\otimes\cdots\otimes F_{w_p}(u_{\ell_p\Lambda_{i_p}})\cdots\big)\Big)$$
$$= \subseteq B\big((\ell_1-\ell_2)\Lambda_{i_1}\big)\otimes B\big((\ell_2-\ell_3)\Lambda_{i_2}\big)\otimes\cdots\otimes B(\ell_p\Lambda_{i_p}).$$

Then the classical limit of $\mathcal{S}$ becomes a $\mathfrak{g}_0\otimes\mathbb{C}[t]$-module. By construction, we have

$$\operatorname{ch}\mathcal{S} = D_{w_1}\Big(e^{(\ell_1-\ell_2)\Lambda_{i_1}}\cdot D_{w_2}\big(e^{(\ell_2-\ell_3)\Lambda_{i_2}}\cdots D_{w_p}(e^{\ell_p\Lambda_{i_p}})\cdots\big)\Big).$$

sketch of the proof.) Let $V(\Lambda)$ denote the irreducible highest weight $U_q(\mathfrak{g})$-module. We define $\mathcal{S}$ by the subspace of

$$V\big((\ell_1-\ell_2)\Lambda_{i_1}\big)\otimes V\big((\ell_2-\ell_3)\Lambda_{i_2}\big)\otimes\cdots\otimes V(\ell_p\Lambda_{i_p})$$

corresponding to the subset

$$\begin{aligned} S = {} & F_{w_1}\Big(u_{(\ell_1-\ell_2)\Lambda_{i_1}}\otimes F_{w_2}\big(u_{(\ell_2-\ell_3)\Lambda_{i_2}}\otimes\cdots\otimes F_{w_p}(u_{\ell_p\Lambda_{i_p}})\cdots\big)\Big) \\ = \subseteq {} & B\big((\ell_1-\ell_2)\Lambda_{i_1}\big)\otimes B\big((\ell_2-\ell_3)\Lambda_{i_2}\big)\otimes\cdots\otimes B(\ell_p\Lambda_{i_p}). \end{aligned}$$

Then the classical limit of $\mathcal{S}$ becomes a $\mathfrak{g}_0\otimes\mathbb{C}[t]$-module. By construction, we have

$$\operatorname{ch}\mathcal{S} = D_{w_1}\Big(e^{(\ell_1-\ell_2)\Lambda_{i_1}}\cdot D_{w_2}\big(e^{(\ell_2-\ell_3)\Lambda_{i_2}}\cdots D_{w_p}(e^{\ell_p\Lambda_{i_p}})\cdots\big)\Big).$$

On the other hand, it is proved by Di Francesco and Kedem that there exists a $\mathfrak{g}_0 \otimes \mathbb{C}[t]$-module M such that

$$\mathrm{ch}\, M = \sum_{\mu \in P_0^+} M(B, \mu, q) \mathrm{ch}\, V_0(\mu).$$

Moreover, we can show that

$$\mathcal{S} \cong M.$$

Hence we have

$$\sum_{\mu \in P_0^+} M(B, \mu, q) \mathrm{ch}\, V_0(\mu) = \mathrm{ch}\, M = \mathrm{ch}\, \mathcal{S}$$

$$= D_{w_1}\Big(e^{(\ell_1 - \ell_2)\Lambda_{i_1}} \cdot D_{w_2}(e^{(\ell_2 - \ell_3)\Lambda_{i_2}} \cdots D_{w_p}(e^{\ell_p \Lambda_{i_p}}) \cdots)\Big). \quad \square$$

On the other hand, it is proved by Di Francesco and Kedem that there exists a $\mathfrak{g}_0 \otimes \mathbb{C}[t]$-module M such that

$$\mathrm{ch}\, M = \sum_{\mu \in P_0^+} M(B, \mu, q) \mathrm{ch}\, V_0(\mu).$$

Moreover, we can show that

$$\mathcal{S} \cong M.$$

Hence we have

$$\begin{aligned}\sum_{\mu \in P_0^+} M(B, \mu, q) \mathrm{ch}\, V_0(\mu) &= \mathrm{ch}\, M = \mathrm{ch}\, \mathcal{S} \\ &= D_{w_1}\Big(e^{(\ell_1 - \ell_2)\Lambda_{i_1}} \cdot D_{w_2}(e^{(\ell_2 - \ell_3)\Lambda_{i_2}} \cdots D_{w_p}(e^{\ell_p \Lambda_{i_p}}) \cdots)\Big). \quad \square\end{aligned}$$

On the other hand, it is proved by Di Francesco and Kedem that there exists a $\mathfrak{g}_0 \otimes \mathbb{C}[t]$-module M such that

$$\mathrm{ch}\, M = \sum_{\mu \in P_0^+} M(B, \mu, q)\,\mathrm{ch}\, V_0(\mu).$$

Moreover, we can show that

$$\mathcal{S} \cong M.$$

Hence we have

$$\begin{aligned}\sum_{\mu \in P_0^+} M(B, \mu, q)\,\mathrm{ch}\, V_0(\mu) &= \mathrm{ch}\, M = \mathrm{ch}\, \mathcal{S} \\ &= D_{w_1}\Big(e^{(\ell_1 - \ell_2)\Lambda_{i_1}} \cdot D_{w_2}(e^{(\ell_2-\ell_3)\Lambda_{i_2}} \cdots D_{w_p}(e^{\ell_p \Lambda_{i_p}}) \cdots)\Big). \quad \square\end{aligned}$$

As a consequence, we have:

Corollary

If $\mathfrak{g}$ is $A_n^{(1)}$ or $D_n^{(1)}$, then we have that

$$\sum_{\mu \in P_0^+} X(B, \mu, q) \operatorname{ch} V_0(\mu) = \sum_{\mu \in P_0^+} M(B, \mu, q) \operatorname{ch} V_0(\mu).$$

Hence the $X = M$ conjecture holds in this case.