Classical limits of minimal affinizations and generalized Demazure modules

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Abstract

Problem

Study the structures of finite-dimensional simple modules over a quantum loop algebra $U_q(L\mathfrak{g})$.

Finite dimensional simple modules over $U_q(L\mathfrak{g})$ are quite many. Hence it seems too ambitious to solve this problem in general (at least for now).

In this talk, we concentrate on some distinguished subclass (minimal affinizations).

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How to study?

M: Minimal affinization of $U_q(L\mathfrak{g})$

$\overset{\text{classical limit}}{\Longrightarrow}$ M_1: $U(L\mathfrak{g})$-module ($L\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$)

$\overset{\tau_a^* \circ \text{Res}}{\Longrightarrow}$ $\overline{M}$: $U(\mathfrak{g} \otimes \mathbb{C}[t])$-module (Restricted limit)

◊ $\mathbf{ch}\, M = \mathbf{ch}\, \overline{M}$

$\overline{M}$ is isomorphic to another $U(\mathfrak{g} \otimes \mathbb{C}[t])$-module (generalized Demazure module)

$\Longrightarrow$ obtain $\mathbf{ch}\, \overline{M}$ ($\Leftrightarrow \mathbf{ch}\, M$)

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finite-dimensional $U_q(\mathfrak{g})$-modules

$\mathfrak{g}$: simple Lie algebra, $I = \{1, \ldots, n\}$: index set,
$\{e_i, h_i, f_i \mid i \in I\}$: Chevalley generators,
relations: $[e_i, f_j] = \delta_{ij} h_i$, $[h_i, e_j] = \langle h_i, \alpha_j \rangle e_i$, …, etc.

$U(\mathfrak{g}) \overset{q\text{-analog}}{\Longrightarrow}$ quantized enveloping algebra $U_q(\mathfrak{g})$

$U_q(\mathfrak{g}) := \langle e_i, k_i^{\pm 1}, f_i \mid i \in I \rangle$ (over $\mathbb{C}(q)$)

relations: $[e_i, f_j] = \delta_{ij} \frac{k_i - k_{-i}}{q_i - q_i^{-1}}$ $(q_i = q^{d_i}, d_i = (\alpha_i, \alpha_i)/2)$,

$$k_i e_j k_i^{-1} = q_i^{\langle h_i, \alpha_j \rangle} e_j, \ldots, \text{ etc. } (k_i \approx q_i^{h_i}).$$

In particular, we can take a limit $q \to 1$ (in a suitable sence)

$$U_q(\mathfrak{g}) \overset{q \to 1}{\Longrightarrow} U(\mathfrak{g}) \qquad \text{(classical limit).}$$

Moreover, classical limit is also defined on modules:

$$V_q : U_q(\mathfrak{g})\text{-module} \overset{q \to 1}{\Longrightarrow} V_1 : U(\mathfrak{g})\text{-module}.$$

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P: weight lattice of $\mathfrak{g}$, P_+: dominant integral weights.

We say a $U_q(\mathfrak{g})$-module V is of type $\mathbf{1}$ if

$$V = \bigoplus_{\lambda \in P} V_\lambda, \quad V_\lambda = \{v \in V \mid k_i v = q_i^{\langle h_i, \lambda \rangle} v\}.$$

In this talk, we assume all the $U_q(\mathfrak{g})$-modules are of type $\mathbf{1}$.

Theorem

Similarly as $\mathfrak{g}$-modules, finite-dimensional simple $U_q(\mathfrak{g})$-modules (of type $\mathbf{1}$) are parametrized by P_+. Moreover, for each $\lambda \in P_+$ we have

$$V_q(\lambda): \ U_q(\mathfrak{g})\text{-module} \xrightarrow{q \to 1} V(\lambda): \ U(\mathfrak{g})\text{-module}.$$

In particular, $\mathrm{ch}\, V_q(\lambda) = \mathrm{ch}\, V(\lambda)$.

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finite-dimensional $U_q(L\mathfrak{g})$-modules

$L\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$: loop algebra

relations : $[h_i \otimes t^m, h_j \otimes t^n] = 0,$
$[h_i \otimes t^m, e_j \otimes t^n] = \langle h_i, \alpha_j \rangle e_j \otimes t^{m+n}, \dots,$ etc.

$\overset{q\text{-analog}}{\Longrightarrow}$ quantum loop algebra $U_q(L\mathfrak{g})$

$U_q(L\mathfrak{g}) = \langle e_{i,m}, f_{i,m}, k_i^{\pm 1}, h_{i,m} \mid i, m \rangle$ (over $\mathbb{C}(q)$)

relations : $[h_{i,m}, h_{j,n}] = 0,$

$$[h_{i,m}, e_{j,n}] = \frac{q_i^{m\langle h_i, \alpha_j \rangle} - q_i^{-m\langle h_i, \alpha_j \rangle}}{m(q_i - q_i^{-1})} e_{j,m+n}, \dots, \text{etc.}$$

In particular, $U_q(L\mathfrak{g}) \overset{q \to 1}{\Longrightarrow} U(L\mathfrak{g})$.

$U^+ := \langle e_{i,m} \mid i, m\rangle,\ U^0 := \langle h_{i,m}, k_i^{\pm 1} \mid i, m\rangle,\ U^- := \langle f_{i,m} \mid i, m\rangle$

$U_q(L\mathfrak{g}) = U^- \cdot U^0 \cdot U^+$: triangular decomposition.

Since $U^0 \cong \mathbb{C}(q)[h_{i,m}, k_i^{\pm 1}]$, we can define

for $\Psi \in \left(\bigoplus_{i,m} \mathbb{C}(q) h_{i,m} \oplus \bigoplus_i \mathbb{C}(q) k_i\right)^*$ a Verma-like module

$$M_q(\Psi) = U_q(L\mathfrak{g}) \otimes_{U^0 \cdot U^+} \mathbb{C}(q)_\Psi.$$

Then $M_q(\Psi)$ has a unique simple quotient $V_q(\Psi)$.

For $i \in I$, define $\Phi_i^{\pm}(u) \in U^0[[u^{\pm 1}]]$ by

$$\Phi_i^{\pm}(u) = k_i^{\pm} \exp\left(\pm (q_i - q_i^{-1}) \Sigma_{m=1}^{\infty} h_{i,m} u^{\pm m} \right).$$

Theorem (Chari, Pressley)

$V_q(\Psi)$ is finite-dimensional if and only if there exists $P_i(u) \in \mathbb{C}(q)[u]$ with constant term 1 for each $i \in I$ such that

$$\Psi\left(\Phi_i^{+}(u)\right) = q_i^{deg(P_i)} \frac{P_i(q_i^{-1}u)}{P_i(q_i u)} = \Psi\left(\Phi_i^{-}(u)\right).$$

$$\{\text{f.d. } U_q(L\mathfrak{g})\text{-mod.}\} \overset{1:1}{\Longleftrightarrow} \{I\text{-tuple of } \mathbb{C}(q)\text{-poly. s.t. } P_i(0) = 1\}$$

$$V_q(\mathcal{P}) \Longleftrightarrow \mathcal{P} = (P_1, \ldots, P_n).$$

$U_q(L\mathfrak{g}) \supseteq U_q(\mathfrak{g}) \Rightarrow \mathrm{ch}\, V$ is defined for a $U_q(L\mathfrak{g})$-module V.
Under mild conditions, we can take
$V_q(P) \overset{q\to 1}{\Longrightarrow} V_1(P)$: $U(L\mathfrak{g})$-module.

However $V_1(P)$ is not necessarily simple,
and the structures of $V_1(P)$ themselves are not so easy to understand.

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Definition of minimal affinization

$V_q(\lambda)$: simple $U_q(\mathfrak{g})$-module corresponding to $\lambda \in P_+$.

Definition

$U_q(L\mathfrak{g})$-module V is an affinization of $V_q(\lambda)$

$\overset{def}{\Leftrightarrow} V \cong V_q(\lambda) \oplus \bigoplus_{\mu<\lambda} V_q(\mu)^{\oplus s_\mu}$ as a $U_q(\mathfrak{g})$-module.

For $\lambda = \sum_{i \in I} m_i \varpi_i \in P_+$,

$$\mathcal{P}^\lambda := \{P = (P_1, \ldots, P_n) \mid P_i(0) = 1, \deg P_i = m_i\}.$$

Fact: $P \in \mathcal{P}^\lambda \Leftrightarrow V_q(P)$ is an affinization of $V_q(\lambda)$.

$V_q(P)$ is a minimal affinization

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Definition (Chari)

(i) *Two affinizations V, W of $V_q(\lambda)$ are* *equivalent*

$\overset{\text{def}}{\Longleftrightarrow}$ $V \cong W$ *as* $U_q(\mathfrak{g})$*-modules.*

($[V]$: equivalent class of V)

(ii) *Define a partial order on equivalent classes as follows: Assume*

$$V \cong V_q(\lambda) \oplus \bigoplus_{\mu<\lambda} V_q(\mu)^{\oplus s_\mu(V)}, \quad W \cong V_q(\lambda) \oplus \bigoplus_{\mu<\lambda} V_q(\mu)^{\oplus s_\mu(W)}.$$

Then $[V] \leq [W] \overset{\text{def}}{\Longleftrightarrow}$ *If μ satisfies $s_\mu(V) > s_\mu(W)$, then $\mu < {}^{\exists}\nu < \lambda$ such that $s_\nu(V) < s_\nu(W)$.*

(iii) *V is* *minimal affinization for λ*

$\overset{\text{def}}{\Longleftrightarrow}$ *$[V]$ is minimal among the affinizations of $V_q(\lambda)$.*

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Minimal affinizations for type A

Assume $\mathfrak{g}$ is of type A_n.
For any $a \in \mathbb{C}(q)^*$, $\exists$ an algebra homomorphism

$$\mathrm{ev}_a \colon U_q(L\mathfrak{g}) \to U_q(\mathfrak{g}),$$

which is a q-analog of the following map:

$$\begin{aligned} \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] &\to \mathfrak{g} \\ x \otimes f &\mapsto f(a)x. \end{aligned}$$

$\therefore$ $\mathrm{ev}_a^*(V_q(\lambda))$ is the unique minimal affinization for λ (up to equivalence).

In other types ev_a does not exist
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Theorem (Chari, Chari-Pressley)

$\mathfrak{g}$: $ABCFG$. For each $\lambda \in P_+$, $\exists!$minimal affinization for λ, and $P \in \mathcal{P}^\lambda$ s.t. $[V_q(P)]$ is minimal were explicitly given.

For type DE, the situation becomes more complicated.

Theorem (Chari-Pressley)

$\mathfrak{g}$: DE. $i_0 \in I$: trivalent node, $J_1, J_2, J_3 \subseteq I$ connected subgraphs such that $I = \bigsqcup_{k=1,2,3} J_k \sqcup \{i_0\}$.

For $\lambda = \sum m_i \varpi_i$,

(i) *$\exists!$minimal affinization if $m_i = 0$ $(\forall i \in J_k)$ for some k,*

(ii) *$\#\{$minimal affinizations$\} = 3$ if (i) is not true and $m_{i_0} \neq 0$,*

(iii) *$\#\{$minimal affinizations$\}$ is not uniformly bounded if (i) is not true and $m_{i_0} = 0$. (irregular case)*

For (i) (ii) (regular case), these $P \in \mathcal{P}^\lambda$ were explicitly given.

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Example: Kirillov-Reshetikhin module

When $\lambda = m\varpi_i$, $\exists!$minimal affinization for λ.

Let $a \in \mathbb{C}(q)^*$, and define $P = (P_1, \dots, P_n)$ by

$$P_j = \begin{cases} (1 - au)(1 - aq_i^2 u) \cdots (1 - aq_i^{2(m-1)} u) & \text{if } j = i, \\ 1 & \text{if } j \neq i. \end{cases}$$

$W^{i,m} := V_q(P)$: the unique minimal affinization for λ

(Kirillov-Reshetikhin (KR) module)

KR modules have several good properties:

(i) T-system, Q-system,

(ii) Fermionic character formula,

(iii) having crystal basis.

Minimal affinizations also have good properties?

(cf. extended T-system for B_n by Mukhin-Young).

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Demazure module

$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d$: affine Lie algebra,
$\widehat{\mathfrak{b}} = \mathfrak{b} \oplus \mathbb{C}K \oplus \mathbb{C}d \oplus \mathfrak{g} \otimes t\mathbb{C}[t]$: Borel subalgebra,
$\widehat{V}(\Lambda)$: simple highest weight module of $\widehat{\mathfrak{g}}$ with h.w. $\Lambda \in \widehat{P}_+$.

Let $\xi \in \widehat{P}$.
There exists a unique $\Lambda \in \widehat{P}_+$ and $w \in \widehat{W}$ such that
$\xi = w(\Lambda)$.

Definition

Let $0 \neq v_\xi \in \widehat{V}(\Lambda)_\xi$. The $\widehat{\mathfrak{b}}$-submodule

$$D(\xi) := U(\widehat{\mathfrak{b}})v_\xi \subseteq \widehat{V}(\Lambda)$$

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For a $\widehat{\mathfrak{g}}$-module $\widehat{V}$ and a $\widehat{\mathfrak{b}}$-submodule $D \subseteq \widehat{V}$, we set

$$\mathcal{F}_i D := U(\widehat{\mathfrak{b}} \oplus \mathbb{C} f_i) D \qquad \text{for } i \in \widehat{I} := \{0\} \cup I.$$

In many cases, $\operatorname{ch} \mathcal{F}_i D = \mathcal{D}_i(\operatorname{ch} D)$ follows where

$$\mathcal{D}_i(f) := \frac{f - e^{-\alpha_i} s_i(f)}{1 - e^{-\alpha_i}} \qquad \text{(Demazure operator).}$$

If $\xi(h_i) \geq 0$, we have

$$\mathcal{F}_i D(\xi) = U(\widehat{\mathfrak{b}} \oplus \mathbb{C} f_i) v_\xi = U(\widehat{\mathfrak{b}}) v_{s_i \xi} = D(s_i \xi).$$

Hence if $\xi = w(\Lambda)$ and $w = s_{i_1} \cdots s_{i_k}$ is reduced,

$$\operatorname{ch} D(\xi) = \operatorname{ch} \mathcal{F}_{i_1} \cdots \mathcal{F}_{i_k} \mathbb{C} v_\Lambda = \mathcal{D}_{i_1} \cdots \mathcal{D}_{i_k}(e^\Lambda).$$

character formular for $D(\xi)$

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Restricted limit

M: Minimal affinization ($U_q(L\mathfrak{g})$-module)

$\overset{\text{classical limit}}{\Longrightarrow}$ M_1: $L\mathfrak{g}(= \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}])$-module

Regard M_1 as a $\mathfrak{g}[t] := \mathfrak{g} \otimes \mathbb{C}[t]$-module by restriction.

There exists $a \in \mathbb{C}$ such that

$$\mathfrak{g} \otimes (t + a)^N M_1 = 0 \qquad \text{for } N >> 0.$$

Define $\tau_a : \mathfrak{g}[t] \to \mathfrak{g}[t]$ by $\tau_a(g \otimes t^n) = g \otimes (t + a)^n$, and

$$\bar{M} := \tau_a^*(M_1) \qquad (\text{Restricted limit}).$$

$\bar{M}$ is a $\mathbb{Z}$-graded $\mathfrak{g}[t]$-module. We have

$$\mathrm{ch}\, M = \mathrm{ch}\, \bar{M}.$$

$\Lambda_0 \in \widehat{P}_+$: fundamental weight of $\widehat{\mathfrak{g}}$,

$\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$, $\quad w_0 \in W$: longest element,

$t_i := (\alpha_i, \alpha_i)/2$ for $i \in I$ (normalized by $(\text{long}, \text{long}) = 2$),

$\overline{W}^{i,m}$: Restricted limit of the KR module $W^{i,m}$.

Theorem (Chari, Chari-Moura, Di Francesco-Kedem)

(i) $\overline{W}^{i,m}$ *is a cyclic* $\mathfrak{g}[t]$*-module with defining relations*

$$\mathfrak{n}_+[t]v = 0, \quad h \otimes t^n v = m\delta_{n0}\varpi_i(h), \quad t^2\mathfrak{n}_-[t]v = 0,$$
$$f_i^{m+1}v = f_i \otimes tv = 0, \quad f_j v = 0 \ (j \neq i).$$

(ii)

$$\overline{W}^{i,m} \cong D(mw_0(\varpi_i) + \lceil mt_i \rceil \Lambda_0),$$

where r.h.s extends to a $\mathfrak{g}[t]$*-module.*

KR module case: Motivation of Main result

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Main results

Assume that M_λ is a minimal affinization for $\lambda = \sum_{i\in I} m_i\varpi_i$.

Theorem

(i) *When $\mathfrak{g}$ is B_n or C_n, $\bar{M}_\lambda$ is a cyclic $\mathfrak{g}[t]$-module with defining relations*

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where $\Delta_+^{(1)} = \{\sum_{i\in I} k_i\alpha_i \mid k_i \le 1\} \subseteq \Delta_+$.

(ii) *When $\mathfrak{g}$ is B_n, $\bar{M}_\lambda$ is isomorphic to the submodule of*

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A similar result of (ii) also holds for C_n.

However, we need to modify the weights of Demazure modules so that the sum of coefficients become even.

Ex. $n = 4$, $\lambda = 8\varpi_1 + 6\varpi_2 + 5\varpi_3 + 5\varpi_4$.

$\bar{M}_\lambda \cong$ the submodule of

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When $\mathfrak{g}$ is D_n and $\#\{\text{min. aff.}\} = 1$ or 3, similar results hold. (They are formulated case by case, and here omit the detail.)

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Corollaries

From the theorem, we obtain two corollaries.

First, let us consider the limit $\lambda \to \infty$ of $\bar{M}_\lambda$.

Then the relations $f_i^{m_i+1} v = 0$ in (i) vanish, and we have

$$\text{``}\bar{M}_\lambda \overset{\lambda\to\infty}{\to} U\Big(\mathfrak{n}_- \oplus \bigoplus_{\alpha \notin \Delta_+^{(1)}} (f_\alpha \otimes t)\Big)\text{''}.$$

Corollary

When $\mathfrak{g}$ is B_n or C_n, we have

$$\lim_{\lambda\to\infty} e^{-\lambda} \mathrm{ch}\, \bar{M}_\lambda = \prod_{\alpha \in \Delta_+} \frac{1}{1-e^{\alpha}} \cdot \prod_{\alpha \notin \Delta_+^{(1)}} \frac{1}{1-e^{\alpha}}.$$

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For simplicity, assume $\mathfrak{g}$ is B_n.

τ: diagram auto. changing the nodes 0 and 1.

It follows that

$$\begin{aligned}\text{the submodule of } & D(m_1 w_0(\varpi_1) + \lceil m_1 t_1 \rceil \Lambda_0) \otimes \\ & \cdots \otimes D(m_n w_0(\varpi_n) + \lceil m_n t_n \rceil \Lambda_0) \\ \cong & \mathcal{F}_{w_0} \tau^* \mathcal{F}_{[1,n-1]}(\mathbb{C}_{m_1\Lambda_0} \otimes \tau^* \mathcal{F}_{[1,n-1]}(\mathbb{C}_{m_2\Lambda_0} \otimes \\ & \cdots \otimes \tau^* \mathcal{F}_{[1,n-1]}(\mathbb{C}_{\lceil m_n/2 \rceil \Lambda_0 + a\Lambda_m}) \cdots))\end{aligned}$$

where $\mathcal{F}_{[1,n-1]} := \mathcal{F}_1 \mathcal{F}_2 \cdots \mathcal{F}_{n-1}$, $a = 0$ if m_n is even and $a = 1$ otherwise.

Corollary

$$\begin{aligned}\mathrm{ch}\, \bar{M}_\lambda = & \mathcal{D}_{w_0} \tau \mathcal{D}_{[1,n-1]}(e^{m_1\Lambda_0} \cdot \tau \mathcal{D}_{[1,n-1]}(e^{m_2\Lambda_0} \\ & \cdots \tau \mathcal{D}_{[1,n-1]}(e^{\lceil m_n/2 \rceil \Lambda_0 + a\Lambda_m}) \cdots)).\end{aligned}$$

brief sketch of the proof of main theorem

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$R(\lambda)$: $\mathfrak{g}[t]$-module in Theorem (i),

$T(\lambda)$: $\mathfrak{g}[t]$-module in Theorem (ii).

goal: $R(\lambda) \cong \bar{M}_\lambda \cong T(\lambda)$.

∘Step 1: Prove $R(\lambda) \twoheadrightarrow \bar{M}_\lambda$ by checking $\bar{M}_\lambda$ satisfies the relations of $R(\lambda)$.

∘Step 2: Prove $\bar{M}_\lambda \twoheadrightarrow T(\lambda)$ as follows:

$$(W^{1,m_1} \otimes \cdots \otimes W^{n,m_n})^* \xrightarrow{\exists} M_\lambda^* \Rightarrow M_\lambda \xrightarrow{\exists} W^{1,m_1} \otimes \cdots \otimes W^{n,m_n}$$

$$\overset{q\to 1}{\Rightarrow} \bar{M}_\lambda \xrightarrow{\exists}$$

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Recall that

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Using this, determin the defining relations of $T(\lambda)$ recursively.

From this, $T(\lambda) \twoheadrightarrow R(\lambda)$ follows.

Thank you for your attention!

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