$U_q(\mathcal{L}\mathfrak{g})$ 加群に対するテンソル積と古典極限を取る操作の非可換性について

直井克之 (Katsuyuki Naoi)

東京農工大学 (Tokyo University of Agriculture and Technology)

日本数学会 2017 年度年会

2017 年 3 月 26 日

Notation

$\mathfrak{g}$: simple Lie algebra/$\mathbb{C}$ $\quad (\mathrm{rank}\,\mathfrak{g} = n)$,

$\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$: triangular dec.,

$\mathcal{L}\mathfrak{g} := \mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}]$: loop algebra $\quad ([X \otimes f, Y \otimes g] = [X, Y] \otimes fg)$,

$U_q(\mathcal{L}\mathfrak{g})$: quantum loop algebra/$\mathbb{C}(q)$ $\quad \overset{q\text{-deform.}}{\Leftarrow} \quad U(\mathcal{L}\mathfrak{g})$

V: f.d. $U_q(\mathcal{L}\mathfrak{g})$-mod. $\quad \overset{\text{“}\lim_{q\to 1}\text{”}}{\Rightarrow} \quad \overline{V}$: f.d. $\mathcal{L}\mathfrak{g}$-mod. (classical limit)

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Precise definition of classical limits

$$\mathcal{A} = \left\{ f(q)/g(q) \in \mathbb{C}(q) \mid g(1) \neq 0 \right\} \subseteq \mathbb{C}(q),$$

$U_q(\mathcal{L}\mathfrak{g}) \supseteq U_{\mathcal{A}}(\mathcal{L}\mathfrak{g}) := \langle E_i, K_i^{\pm 1}, F_i \rangle_{\mathcal{A}\text{-alg.}}$,

Fact $\quad U_{\mathcal{A}}(\mathcal{L}\mathfrak{g}) \otimes_{\mathcal{A}} \mathbb{C} \twoheadrightarrow U(\mathcal{L}\mathfrak{g})$.

V: fin. dim. $U_q(\mathcal{L}\mathfrak{g})$-mod. Assume that V is ℓ-h.w. module with ℓ-h.w.vector v (i.e. $U_q(\mathcal{L}\mathfrak{n}_+)v = 0$, $U_q(\mathcal{L}\mathfrak{h})v = \mathbb{C}v$, $U_q(\mathcal{L}\mathfrak{n}_-)v = V$), and further assume that $U_{\mathcal{A}}(\mathcal{L}\mathfrak{g})v \subseteq V$ is an $\mathcal{A}$-lattice of V

($\Leftrightarrow$ mild condition on the ℓ-h.w.)

Definition (Chari-Pressley, 01)

$\overline{V} := U_{\mathcal{A}}(\mathcal{L}\mathfrak{g})v \otimes_{\mathcal{A}} \mathbb{C}$: classical limit $\quad \leftarrow \mathcal{L}\mathfrak{g}$-mod.

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problem

$V_1, \ldots, V_p$: f.d. $U_q(\mathcal{L}\mathfrak{g})$-mod. ($v_k \in V_k$: ℓ-h.w. vector)

$\overline{V_1 \otimes \cdots \otimes V_p} = U_{\mathcal{A}}(\mathcal{L}\mathfrak{g})(v_1 \otimes \cdots \otimes v_p) \otimes_{\mathcal{A}} \mathbb{C}$

$\updownarrow$ not necessarily isomorphic

$\overline{V_1} \otimes \cdots \otimes \overline{V_p} \cong \left(U_{\mathcal{A}}(\mathcal{L}\mathfrak{g})v_1 \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} U_{\mathcal{A}}(\mathcal{L}\mathfrak{g})v_p\right) \otimes_{\mathcal{A}} \mathbb{C}$

Q. Can we construct $\overline{V_1 \otimes \cdots \otimes V_p}$ from $\overline{V_1}, \ldots, \overline{V_p}$?

We solved this question affirmatively for Kirillov-Reshetikhin modules.

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Notions for the statement of the main theorem

Kirillov-Reshetikhin modules

$W^{i,\ell}(a)$ $(1 \leq i \leq n, \ell \in \mathbb{Z}_{>0}, a \in \mathbb{C}(q)^{\times})$: Kirillov-Reshetikhin mod.

(a distinguished family of f.d. simple $U_q(\mathcal{L}\mathfrak{g})$-mod.)

graded limits

$\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t] \subseteq \mathcal{L}\mathfrak{g}$: current algebra,

φ_c $(c \in \mathbb{C})$: auto. on $\mathfrak{g}[t]$ defined by $\varphi_c(X \otimes f(t)) = X \otimes f(t+c)$.

Fact Assume $a \in \mathcal{A}^{\times}$, and set $c = a(1) \in \mathbb{C}^{\times}$.

Then $\varphi_{-c}^{*}\overline{W^{i,\ell}(a)}$ is $\mathbb{Z}$-graded $\mathfrak{g}[t]$-mod., and independent of a.

$\rightsquigarrow$ write $W^{i,\ell} := \varphi_{-c}^{*}\overline{W^{i,\ell}(a)}$: graded limit of $W^{i,\ell}(a)$.

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fusion product [Feigin-Loktev, '99]

$M_1, \dots, M_p$: cyclic $\mathbb{Z}$-graded $\mathfrak{g}[t]$-mod. ($v_k \in M_k$: generator)

Take $c_1, \dots, c_p \in \mathbb{C}$ (pairwise distinct),

and set $M := \varphi_{c_1}^* M_1 \otimes \cdots \otimes \varphi_{c_p}^* M_p$.

Fact M is generated by $v_1 \otimes \cdots \otimes v_p$ (though not $\mathbb{Z}$-graded).

(Note that usual tensor product $M_1 \otimes \cdots \otimes M_p$ is not cyclic)

Define a filtration $F_{-1}(M) = 0 \subseteq F_0(M) \subseteq \cdots \subseteq F_N(M) = M$ ($N \gg 0$) by $F_k(M) = U(\mathfrak{g}[t])_{\leq k}(v_1 \otimes \cdots \otimes v_p)$, and take the associated graded $\bigoplus_k F_k(M)/F_{k-1}(M) \leftarrow \mathbb{Z}$-graded $\mathfrak{g}[t]$-mod.

This is called the fusion product of $M_1, \dots, M_p$, and denoted by $M_1 * \cdots * M_p$.

fusion prod. = deformed $\otimes$ preserving cyclicity and $\mathbb{Z}$-gradedness fusion product [Feigin-Loktev, '99]

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Main Theorem

Theorem (N)

Assume that a given tensor prod. $W^{i_1,\ell_1}(a_1) \otimes \cdots \otimes W^{i_p,\ell_p}(a_p)$ of KR mod. has its classical limit ($^{\exists}$sufficient conditions for i_k, ℓ_k, a_k).

(i) If $a_1(1) = \cdots = a_p(1) (=: c)$, then we have a $\mathfrak{g}[t]$-mod. isom.

$$\overline{W^{i_1,\ell_1}(a_1) \otimes \cdots \otimes W^{i_p,\ell_p}(a_p)} \cong \varphi_c^*(W^{i_1,\ell_1} * \cdots * W^{i_p,\ell_p}).$$

(ii) In the general case, we have a $\mathfrak{g}[t]$-mod. isom.

$$\overline{W^{i_1,\ell_1}(a_1) \otimes \cdots \otimes W^{i_p,\ell_p}(a_p)} \cong \bigotimes_{c \in \mathbb{C}^\times} \varphi_c^* \Big(\mathop{\Large *}_{k;\, a_k(1)=c} W^{i_k,\ell_k} \Big).$$

<u>recall</u> φ_c: auto. on $\mathfrak{g}[t]$ defined by $\varphi_c(X \otimes f(t)) = X \otimes f(t+c)$,
$W^{i,\ell}$: the graded limit of $W^{i,\ell}(a)$, $*$: fusion product