An approach to the $X = M$ conjecture using current algebras

Katsuyuki Naoi (Kavli IPMU)

Bethe Ansatz, Quantum Groups and Beyond

March 7th, 2013

Plan

1. What is the $X = M$ conjecture?

   1-dimensional sum X = fermionic formula M $\in \mathbb{Z}[u^{\pm 1}]$

   (crystal basis theory) (Bethe Ansatz)

   - Definitions of X, M (with histrical background)

2. Proof in type AD (and partially in BC)

   ◊ Use representations of a current algebra $\mathfrak{g} \otimes \mathbb{C}[t]$

   - graded limits of KR modules
   - fusion products
   - generalized Demazure modules

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$$\left(p_j = L - 2\sum_j \min\{i,j\} m_j \quad \text{(vacancy number)} \right)$$

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The formula can be generalized in type A_n:

$$\left[V(\mu_1\varpi_1)\otimes\cdots\otimes V(\mu_p\varpi_1) : V(\lambda)\right] = \sum_{\substack{m=\{m_j^{(a)}\}_{\substack{1\le a\le n\\ j\ge 1}} \\ \text{s.t. } \cdots}} \prod_{a,j} \binom{p_j^{(a)} + m_j^{(a)}}{m_j^{(a)}}.$$

Moreover, the graded version also holds!

$$K_{\lambda,\mu}(u) = \sum_m u^{c(m)} \prod_{a,j} \begin{bmatrix} p_j^{(a)} + m_j^{(a)} \\ m_j^{(a)} \end{bmatrix}_u \in \mathbb{Z}_{\ge 0}[u^{\pm 1}].$$

$$\begin{pmatrix} K_{\lambda,\mu}(u) : \text{Kostka polynomial} \\ c(m) \in \mathbb{Z} : \text{charge} \end{pmatrix}$$

Since $K_{\lambda,\mu}(1) = \left[V(\mu_1\varpi_1)\otimes\cdots\otimes V(\mu_p\varpi_1) : V(\lambda)\right]$,

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This follows from a bijection (KKR bijection)

$SST(\lambda, \mu)$: s.s. tableaux $\overset{1:1}{\leftrightarrow}$ $RC(\mu, \lambda)$: rigged configurations

preserving their gradings.

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Can this be generalized in the other types?

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Nongraded version for general type

$\mathfrak{g}$: simple Lie algebra of rank n, $\quad I = \{1, \dots, n\}$,

$U'_q(\hat{\mathfrak{g}})$: quantum affine algebra (without a degree operator),

$W^{r,\ell}$: Kirillov-Reshetikhin (KR) module ($r \in I$, $\ell \in \mathbb{Z}_{>0}$)

(a family of f.d. simple $U'_q(\hat{\mathfrak{g}})$-modules).

Then (nongraded) fermionic formula follows in general:

Theorem ([Nakajima,'03], [Hernandez,'06], [DiFrancesco,Kedem,'08])

$$\underbrace{\left[W^{r_1,\ell_1} \otimes \cdots \otimes W^{r_p,\ell_p} : V(\lambda)\right]}_{(U_q(\mathfrak{g})\text{-multiplicity})} = \sum_{m=\{m_i^{(a)}\}} \prod_{a,j} \binom{p_j^{(a)} + m_j^{(a)}}{m_j^{(a)}}$$

(When $\mathfrak{g} = A_n$, $W^{r,\ell} \cong V(\ell\varpi_r)$ as a $U_q(\mathfrak{g})$-module.)

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In type A_n, Kostka polynomial $K_{\mu,\lambda}(u)$ is expressed as a generating function of semistandard tableaux.

In general type, we use crystal basis instead.

Theorem ([Kashiwara, 04], [Okado,Schilling,'08])

A KR module $W^{r,\ell}$ has a crystal basis if

- $\hat{\mathfrak{g}}$: general type, $\ell = 1$, $^{\forall}r$,
- $\hat{\mathfrak{g}}$: nonexceptional type, $^{\forall}r$, $^{\forall}\ell$.

$B^{r,\ell}$: the corresponding crystal graph (KR crystal),

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Definition of the 1-dimensional sum $X(W, \lambda, u)$

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Energy function $D : B \to \mathbb{Z}$ is defined combinatorially.

Then we define the 1-dimensional sum $X(W, \lambda, u)$ by

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which satisfies $X(W, \lambda, 1) = [W : V(\lambda)]$ as required.

Theorem (Nakayashiki, Yamada, ’97)

In type A_n, we have

$$K_{\lambda,\mu}(q) = X(W^\mu, \lambda, u),$$

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for all $W = W^{r_1,\ell_1} \otimes \cdots \otimes W^{r_p,\ell_p}$ and λ.

The conjecture has been proved in the following cases.

- $\hat{\mathfrak{g}} = A_n^{(1)}$, ${}^\forall W$ [Kirillov, Schilling, Shimozono, 2002], (KKR bijection)
- $\hat{\mathfrak{g}}$: nonexceptional type, $\operatorname{rk} \mathfrak{g} \gg 0$ [Lecouvey,Okado,Shimozono,'10],[Okado,Sakamoto,'10], (proved $X = K$ and $M = K$ respectively),
- $\hat{\mathfrak{g}}$: non-twisted, W: $\ell_i = 1$ for all i [N],
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$$W = W^{r_1,\ell_1} \otimes \cdots \otimes W^{r_p,\ell_p}, \quad B = B^{r_1,\ell_1} \otimes \cdots \otimes B^{r_p,\ell_p},$$

fermionic formula $M(W, \lambda, u)$

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graded multiplicity $[L : V(\lambda)]_{u^{-1}}$

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$\mathfrak{g}[t] := \mathfrak{g} \otimes \mathbb{C}[t]$: current algebra ($\mathbb{Z}_{\geq 0}$-graded Lie algebra),

$\mathfrak{g}[t] \curvearrowleft \tau_a \ (a \in \mathbb{C})$: $\tau_a(x \otimes t^k) = x \otimes (t+a)^k$,

$L^{r,\ell}$: graded limit of a KR module $W^{r,\ell}$ constructed as follows:

$W^{r,\ell}$: $U'_q(\hat{\mathfrak{g}})$-module $\overset{q\to 1}{\to}$ $\overline{W^{r,\ell}}$: $\mathfrak{g} \otimes \mathbb{C}[t,t^{-1}]$-module

$\overset{\text{restriction}}{\to}$ $\overline{W^{r,\ell}}$: $\mathfrak{g}[t]$-module

$\overset{\tau_a^*}{\to}$ $L^{r,\ell} := \tau_a^*(\overline{W^{r,\ell}})$: $\mathbb{Z}$-graded $\mathfrak{g}[t]$-module ($^\exists a \in \mathbb{C}$)

- $L^{r,\ell}$ can also be defined in terms of generators and relations [Chari, Moura, ’06].

Expression of $M(W, \mu, q)$ by characters

$\mathfrak{g}[t] := \mathfrak{g} \otimes \mathbb{C}[t]$: current algebra ($\mathbb{Z}_{\geq 0}$-graded Lie algebra),

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$W^{r,\ell}$: $U'_q(\hat{\mathfrak{g}})$-module $\overset{q \to 1}{\to}$ $\overline{W^{r,\ell}}$: $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$-module

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$\overset{\tau_a^*}{\to}$ $L^{r,\ell} := \tau_a^*(\overline{W^{r,\ell}})$: $\mathbb{Z}$-graded $\mathfrak{g}[t]$-module ($^\exists a \in \mathbb{C}$)

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$$L = L^{r_1,\ell_1} * \cdots * L^{r_p,\ell_p} := \mathrm{gr}(\tau_{c_1}^*(L^{r_1,\ell_1}) \otimes \cdots \otimes \tau_{c_p}^*(L^{r_p,\ell_p})),$$

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$$M(W, \lambda, u) = [L : V(\lambda)]_{u^{-1}} := \sum_j u^{-j}[L_j : V(\lambda)].$$

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Plan of the proof

fermionic formula $M(W, \lambda, u)$

$\|$ [Ardonne, Kedem]

graded multiplicity $[L : V(\lambda)]_{u^{-1}}$

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$L \cong D(\vec{w}, \vec{\Lambda})$: generalized Demazure module

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Generalized Demazure module

$\hat{\mathfrak{g}} := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d$: affine Lie algebra $\supseteq \mathfrak{g}[t]$,

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V: $\hat{\mathfrak{g}}$-mod $\supseteq D$: $\hat{\mathfrak{b}}$-submod, $\mathcal{F}_i(D) := U(\hat{\mathfrak{b}} \oplus \mathbb{C} f_i) D \subseteq V$,

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(Generalized Demazure module)

- For some $\vec{w}, \vec{\Lambda}$, $D(\vec{w}, \vec{\Lambda})$ extends to a $\mathfrak{g}[t]$-module.
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For an index $r \in I$, set

$$c_r := \begin{cases} 1 & (\alpha_r : \text{long root}), \\ 2 & (\mathfrak{g} : BCF,\ \alpha_r : \text{ short root}), \\ 3 & (\mathfrak{g} : G,\ \alpha_r : \text{ short root}). \end{cases}$$

Note that $c_r = 1$ when $\mathfrak{g} = ADE$.

Theorem ([Chari, Moura, '06], [Fourier, Littelmann, '07])

(i) For $r \in I$ and $\ell \in \mathbb{Z}_{>0}$,

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Corollary

$$[L^{r_1,\ell_1} * \cdots * L^{r_p,\ell_p} : V(\lambda)]_{u^{-1}} = [D(\vec{w}, \vec{\Lambda}) : V(\lambda)]_{u^{-1}}$$

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Plan of the proof

graded multiplicity $[L : V(\lambda)]_{u^{-1}}$

$\parallel$

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$\mathbf{1}$-dimensional sum $X(W, \lambda, u)$

Plan of the proof

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$B \xrightarrow{\sim} B(\vec{w}, \vec{\Lambda})$: "crystal analog" of $D(\vec{w}, \vec{\Lambda})$

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$V_q(\Lambda)$: h.w. simple $U_q(\hat{\mathfrak{g}})$-module,

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For B: $U_q(\hat{\mathfrak{g}})$-crystal basis, $D \subseteq B$: subset,

set $\tilde{\mathcal{F}}_i(D) := \{\tilde{f}_i^k(b) \mid k \geq 0,\ b \in D\} \subseteq B$,

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- $B(w_1, \Lambda_1)$ is called a Demazure crystal [Kashiwara, 93].

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$$\mathrm{ch}\ D(\vec{w}, \vec{\Lambda}) = \sum_{b \in B(\vec{w}, \vec{\Lambda})} e(\mathrm{wt}(b)).$$

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$B^{r,\ell}$: perfect $\Leftrightarrow c_r|\ell$ ([Fourier, Okado, Schilling, '10]).

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the energy fct D expresses the action of a degree op. d,

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Relations between $B(\vec{w}, \vec{\Lambda})$ and KR crystal $B^{r,\ell}$

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Assume $\mathfrak{g}$ is a classical type.

$B^{r,\ell}$: perfect $\Leftrightarrow c_r|\ell$ ([Fourier, Okado, Schilling, '10]).

$$\Phi: \quad B(\ell\Lambda_0) \otimes B^{r_1,c_{r_1}\ell} \otimes B^{r_2,c_{r_2}\ell} \otimes \cdots \otimes B^{r_p,c_{r_p}\ell}$$
$$\cong \; B({}^{\exists}\Lambda') \otimes B^{r_2,c_{r_2}\ell} \otimes \cdots \otimes B^{r_p,c_{r_p}\ell} \cong \cdots \cong B({}^{\exists}\Lambda)$$

[Kang,Kashiwara,Misra,Miwa,Nakashima,Nakayashiki,'92].

Theorem (Schilling, Tingley, '12)

$$\Phi(u_{\ell\Lambda_0} \otimes B^{r_1,c_{r_1}\ell} \otimes \cdots \otimes B^{r_p,c_{r_p}\ell}) = B({}^{\exists}w, \Lambda), \text{ and}$$

the energy fct D expresses the action of a degree op. d,

i.e. $\langle d, \mathrm{wt}\,\Psi(u_{\ell\Lambda_0} \otimes b)\rangle = -D(b)$

- The elements w, Λ are the same with ones in

$$L^{r_1,c_{r_1}\ell} * \cdots * L^{r_p,c_{r_p}\ell} \cong {}^{\exists}D(w, \Lambda).$$

The above theorem can be generalized as follows:

Theorem (N)

Let $\ell := \max\{\ell_1, \dots, \ell_p\}$. Then we have

$$^\exists\Psi : B(\ell\Lambda_0) \otimes B^{r_1, c_{r_1}\ell_1} \otimes \cdots \otimes B^{r_p, c_{r_p}\ell_p} \hookrightarrow B(\Lambda^1) \otimes \cdots \otimes B(\Lambda^p)$$

s.t. $\Psi(u_{\ell\Lambda_0} \otimes B^{r_1, c_{r_1}\ell_1} \otimes \cdots \otimes B^{r_p, c_{r_p}\ell_p}) = B(^\exists\vec{w}, \vec{\Lambda})$.

Moreover, we have

$$\langle \mathrm{wt}\, \Psi(u_{\ell\Lambda_0} \otimes b), d\rangle = -D(b)$$

- The $\vec{w}, \vec{\Lambda}$ are the same with that appeared in Step 2.

$$\Rightarrow \quad X(W, \lambda, u) = \sum_{\substack{b \in B \\ \tilde{e}_i(b)=0, \mathrm{wt}(b)=\lambda}} u^{D(b)} = \sum_{\substack{b \in B(\vec{w}, \vec{\Lambda}) \\ \tilde{e}_i(b)=0, \mathrm{wt}(b)=\lambda}} u^{-\langle d, \mathrm{wt}(b)\rangle}$$

$$\overset{\text{Prop}}{=} [D(\vec{w}, \vec{\Lambda}) : V(\lambda)]_{u^{-1}}.$$

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Corollary

$\mathfrak{g}$: a classical type, $r_1, \ldots, r_p \in I$, $\ell_1, \ldots, \ell_p \in \mathbb{Z}_{>0}$,
$W := W^{r_1, c_{r_1}\ell_1} \otimes \cdots \otimes W^{r_p, c_{r_p}\ell_p}$. Then we have

$$X(W, \lambda, u) = M(W, \lambda, u).$$

Note that c_r is 1 for all r when $\mathfrak{g}$ is of type AD.

Thank you for your attention.

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