Equivalence between module categories over quiver Hecke algebras and Hernandez-Leclerc's categories

Katsuyuki Naoi

Tokyo University of Agriculture and Technology

Conference on Algebraic Representation Theory 2021,

November 29, 2021

based on a paper in Adv. in Math. 389, 8 (2021), arXiv:2101.03573

Summary of today's result

$$\begin{pmatrix}\text{f.d. mod. over a}\\ \text{quantum affine alg. } U_q'(\mathfrak{g})\end{pmatrix}$$

$$\bigcup$$

$$\begin{pmatrix}\text{f.d. mod. over}\\ \text{quiver Hecke algebras } R(\beta)\end{pmatrix} \xrightarrow[\text{by Kang–Kashiwara–Kim}]{\mathcal{F}\text{: gen. Q-affine SW duality functor}} \begin{pmatrix}\text{Hernandez–Leclerc's}\\ \text{subcategory } \mathcal{C}_{\mathbf{Q}}\end{pmatrix}$$

Theorem ([N])

In a general affine type, $\mathcal{F}$ gives an equivalence of two monoidal categories.

In untwisted ADE types, this was previously proved by Fujita.

Summary of today's result

$\left(\begin{array}{c}\text{f.d. mod. over a}\\ \text{quantum affine alg. } U_q'(\mathfrak{g})\end{array}\right)$

$\bigcup$

$\left(\begin{array}{c}\text{f.d. mod. over}\\ \text{quiver Hecke algebras } R(\beta)\end{array}\right) \overset{\substack{\mathcal{F}\text{: gen. Q-affine}\\ \text{SW duality functor}\\ \text{by Kang–Kashiwara–Kim}}}{\Longrightarrow} \left(\begin{array}{c}\text{Hernandez–Leclerc's}\\ \text{subcategory } \mathcal{C}_{\mathbf{Q}}\end{array}\right)$

Theorem ([N])

In a general affine type, $\mathcal{F}$ gives an equivalence of two monoidal categories.

In untwisted ADE types, this was previously proved by Fujita.

Notation $A\text{-Mod}$: cat. of f.g. A-modules

$A\text{-mod}$: cat. of f.d. A-modules

Quiver Hecke algebras $R(\beta)$

Khovanov–Lauda [KL09] and Rouquier [Rou08] defined independently.

Given a Kac-Moody g whose Cartan matrix is A

$\rightsquigarrow R(\beta)$: **quiver Hecke algebras** (family of algebras, $\beta \in Q^+ = \sum_i \mathbb{Z}_{\geq 0}\alpha_i$)

$\circ$ $R(\beta)$ are $\mathbb{Z}$-graded algebras,

$\circ$ $M \in R(\beta)\text{-gmod}$, $M' \in R(\beta')\text{-gmod}$,

$\rightsquigarrow M \circ M' \in R(\beta + \beta')\text{-gmod}$: **convolution product**

Quiver Hecke algebras $R(\beta)$

Khovanov–Lauda [KL09] and Rouquier [Rou08] defined independently.

Given a Kac-Moody $\mathfrak{g}$ whose Cartan matrix is A

$\rightsquigarrow R(\beta)$: **quiver Hecke algebras** (family of algebras, $\beta \in Q^+ = \sum_i \mathbb{Z}_{\geq 0}\alpha_i$)

- $R(\beta)$ are $\mathbb{Z}$-graded algebras,
- $M \in R(\beta)\text{-gmod}$, $M' \in R(\beta')\text{-gmod}$,

$\rightsquigarrow M \circ M' \in R(\beta + \beta')\text{-gmod}$: **convolution product**

Theorem ([KL09],[Rou08])

$\bigoplus_\beta K(R(\beta)\text{-gmod}) \cong U_{\mathbb{Z}}^-(\mathfrak{g})^\vee$: int. form of the dual of the half of $U_q(\mathfrak{g})$ (as $\mathbb{Z}[q^{\pm 1}]$-algebra)

Theorem ([Varagnolo-Vasserot, 11], [Rouquier, 12])

$\mathfrak{g}$: symmetric $\Rightarrow$ the isom. sends simples to the upper global basis.

$$\bigoplus_{\beta} K(R(\beta)\text{-gmod}) \cong U_{\mathbb{Z}}^{-}(\mathrm{g})^{\vee}$$

$$\begin{array}{ccc} \cup & & \cup \\ \{\text{simples}\} & \to & \{\text{upper global basis}\} \end{array}$$

By specializing at $q = 1$, we obtain the following.

Corollary

If g is a simple Lie algebra of type ADE,

(i) $\bigoplus_{\beta} \mathbb{C} \otimes_{\mathbb{Z}} K(R(\beta)\text{-mod}^0) \cong \mathbb{C}[N]$,

where $R(\beta)\text{-mod}^0$: cat. of f.d. mod. on which x_k's act nilpotently

(obtained from graded ones by forgetting the gradings)

$\mathbb{C}[N]$: coordinate ring of the unipotent group associated with g.

(ii) This isom. sends simples to (the specialization of) upper global basis.

$$\bigoplus_\beta K(R(\beta)\text{-gmod}) \cong U_{\mathbb{Z}}^-(\mathrm{g})^\vee$$

$$\cup \qquad\qquad \cup$$

$$\{\text{simples}\} \to \{\text{upper global basis}\}$$

By specializing at $q = 1$, we obtain the following.

Corollary

If g is a simple Lie algebra of type ADE,

(i) $\bigoplus_\beta \mathbb{C} \otimes_{\mathbb{Z}} K(R(\beta)\text{-mod}^0) \cong \mathbb{C}[N]$,

where $R(\beta)\text{-mod}^0$: cat. of f.d. mod. on which x_k's act nilpotently

(obtained from graded ones by forgetting the gradings)

$\mathbb{C}[N]$: coordinate ring of the unipotent group associated with g.

(ii) This isom. sends simples to (the specialization of) upper global basis.

There is another algebra categorifying the same things!

Hernandez–Leclerc's subcategory

[Hernandez–Leclerc, 15]

$\mathfrak{g}$: simple Lie algebra of type ADE, R^+: positive roots of $\mathfrak{g}$

$\hat{\mathfrak{g}}$: untwisted affine Lie alg. assoc. with $\mathfrak{g}$ $\rightsquigarrow U_q'(\widehat{\mathfrak{g}})$: quantum group of $\widehat{\mathfrak{g}}$,

$\mathcal{C}_{\hat{\mathfrak{g}}} := U_q'(\hat{\mathfrak{g}})\text{-mod}$: cat. of f.d. $U_q'(\hat{\mathfrak{g}})$-modules

Hernandez–Leclerc's subcategory

[Hernandez–Leclerc, 15]

$\mathfrak{g}$: simple Lie algebra of type ADE, R^+: positive roots of $\mathfrak{g}$
$\hat{\mathfrak{g}}$: untwisted affine Lie alg. assoc. with $\mathfrak{g}$ $\rightsquigarrow U_q'(\widehat{\mathfrak{g}})$: quantum group of $\widehat{\mathfrak{g}}$,
$\mathcal{C}_{\hat{\mathfrak{g}}} := U_q'(\hat{\mathfrak{g}})\text{-mod}$: cat. of f.d. $U_q'(\hat{\mathfrak{g}})$-modules

They defined a map $Q\colon R^+ \ni \alpha \mapsto V^\alpha \in \mathcal{C}_{\hat{\mathfrak{g}}}$: simple (fundamental) modules (using the theory of Auslander-Reiten quiver)

Hernandez–Leclerc's subcategory

[Hernandez–Leclerc, 15]

$\mathfrak{g}$: simple Lie algebra of type ADE, R^+: positive roots of $\mathfrak{g}$
$\hat{\mathfrak{g}}$: untwisted affine Lie alg. assoc. with $\mathfrak{g}$ $\rightsquigarrow U_q'(\widehat{\mathfrak{g}})$: quantum group of $\widehat{\mathfrak{g}}$,
$\mathcal{C}_{\hat{\mathfrak{g}}} := U_q'(\hat{\mathfrak{g}})\text{-mod}$: cat. of f.d. $U_q'(\hat{\mathfrak{g}})$-modules

They defined a map $Q\colon R^+ \ni \alpha \mapsto V^\alpha \in \mathcal{C}_{\hat{\mathfrak{g}}}$: simple (fundamental) modules (using the theory of Auslander-Reiten quiver)
$\rightsquigarrow \mathcal{C}_{\hat{\mathfrak{g}}} \supseteq \mathcal{C}_Q$: containing V^α $(\alpha \in R^+)$, stable under $\otimes$, ext., subquot..

Hernandez–Leclerc's subcategory

[Hernandez–Leclerc, 15]

$\mathfrak{g}$: simple Lie algebra of type ADE, R^+: positive roots of $\mathfrak{g}$
$\hat{\mathfrak{g}}$: untwisted affine Lie alg. assoc. with $\mathfrak{g}$ $\rightsquigarrow U_q'(\widehat{\mathfrak{g}})$: quantum group of $\widehat{\mathfrak{g}}$,
$\mathcal{C}_{\hat{\mathfrak{g}}} := U_q'(\hat{\mathfrak{g}})\text{-mod}$: cat. of f.d. $U_q'(\hat{\mathfrak{g}})$-modules

They defined a map $Q\colon R^+ \ni \alpha \mapsto V^\alpha \in \mathcal{C}_{\hat{\mathfrak{g}}}$: simple (fundamental) modules (using the theory of Auslander-Reiten quiver)
$\rightsquigarrow \mathcal{C}_{\hat{\mathfrak{g}}} \supseteq \mathcal{C}_Q$: containing V^α $(\alpha \in R^+)$, stable under $\otimes$, ext., subquot..

Theorem

$\mathbb{C} \otimes_{\mathbb{Z}} K(\mathcal{C}_Q) \cong \mathbb{C}[N]$ as a $\mathbb{C}$-algebra, and this sends simples to (the specialization of) upper global basis.

$$\bigoplus_{\beta} \mathbb{C} \otimes_{\mathbb{Z}} K(R(\beta)\text{-mod}^0) \cong \quad \mathbb{C}[N] \quad \cong \mathbb{C} \otimes_{\mathbb{Z}} K(\mathcal{C}_Q)$$

(simples) $\leftrightarrow$ (gl. basis) $\leftrightarrow$ (simples)

Q. Is there a functor between $R(\beta)\text{-mod}^0$ and $\mathcal{C}_Q$ inducing this isomorphism?

$$\bigoplus_{\beta} \mathbb{C} \otimes_{\mathbb{Z}} K(R(\beta)\text{-mod}^0) \cong \quad \mathbb{C}[N] \quad \cong \mathbb{C} \otimes_{\mathbb{Z}} K(\mathcal{C}_Q)$$

(simples) $\leftrightarrow$ (gl. basis) $\leftrightarrow$ (simples)

Q. Is there a functor between $R(\beta)\text{-mod}^0$ and $\mathcal{C}_Q$ inducing this isomorphism?

Type A [Chari–Pressley, Cherednik, Ginzburg–Varagnolo–Vasserot]

$R(\beta)\text{-mod}^0 \risingdotseq H_q^{\mathrm{aff}}(d)\text{-mod}$ (affine Hecke algebra)

$\mathbb{V}^{\otimes d}$: $(U_q'(\widehat{\mathfrak{sl}}_n), H_q^{\mathrm{aff}}(d))$-bimodule

$\Rightarrow H_q^{\mathrm{aff}}(d)\text{-mod} \ni M \mapsto \mathbb{V}^{\otimes d} \otimes_{H_q^{\mathrm{aff}}(d)} M \in \mathcal{C}_{\widehat{\mathfrak{sl}}_n}$

(**quantum affine Schur–Weyl duality functor**)

Kang–Kashiwara–Kim's construction of functors

[KKK18]: construction of functors in general setting

[KKK15]: application of the results in [KKK18] to HL subcategories
(giving an answer to the previous Q.)

Kang–Kashiwara–Kim's construction of functors

[KKK18]: construction of functors in general setting

[KKK15]: application of the results in [KKK18] to HL subcategories
(giving an answer to the previous Q.)

Review on [KKK18] (Given data $\{V_i\} \rightsquigarrow \mathcal{F}\colon \bigoplus_\beta R(\beta)\text{-mod}^0 \to \mathcal{C}_\mathfrak{g}$)

Kang–Kashiwara–Kim's construction of functors

[KKK18]: construction of functors in general setting

[KKK15]: application of the results in [KKK18] to HL subcategories (giving an answer to the previous Q.)

Review on [KKK18] (Given data $\{V_i\} \rightsquigarrow \mathcal{F}\colon \bigoplus_\beta R(\beta)\text{-mod}^0 \to \mathcal{C}_\mathfrak{g}$)

$U_q'(\mathfrak{g})$: quantum affine algebra of a general affine type, $\mathcal{C}_\mathfrak{g} := U_q'(\mathfrak{g})\text{-mod}$

Given a family of real simple modules $\{V_i\}_{i\in J} \in \mathcal{C}_\mathfrak{g}$

Kang–Kashiwara–Kim's construction of functors

[KKK18]: construction of functors in general setting

[KKK15]: application of the results in [KKK18] to HL subcategories (giving an answer to the previous Q.)

Review on [KKK18] (Given data $\{V_i\} \rightsquigarrow \mathcal{F}\colon \bigoplus_\beta R(\beta)\text{-mod}^0 \to \mathcal{C}_\mathfrak{g}$)

$U'_q(\mathfrak{g})$: quantum affine algebra of a general affine type, $\mathcal{C}_\mathfrak{g} := U'_q(\mathfrak{g})\text{-mod}$

Given a family of real simple modules $\{V_i\}_{i\in J} \in \mathcal{C}_\mathfrak{g}$

$\rightsquigarrow$ define a Cartan matrix $A = (a_{ij})_{i,j\in J}$ by

$$a_{ij} = \begin{cases} 2 & (i=j), \\ -b_{ij} - b_{ji} & (i \neq j), \end{cases}$$ where

$$b_{ij} = \big(\text{deg. of pole of } V_i \otimes (V_j[z^{\pm 1}]) \xrightarrow{R^{\mathrm{norm}}} (V_j(z)) \otimes V_i \text{ at } z = 1\big).$$

$\rightsquigarrow \{R(\beta)\}_{\beta\in Q^+}$: quiver Hecke algebras assoc. with A

Then we construct a $(U_q'(\mathfrak{g}), R(\beta))$-bimodule as follows.

V_i $(i \in J) \rightsquigarrow \widehat{V}_i = V_i[\![w]\!]$: a completed affinization $(U_q'(\mathfrak{g})$-module)

For $\beta \in Q^+$, $\widehat{V}^{\otimes\beta} = \bigoplus_{\alpha_{i_1}+\cdots+\alpha_{i_p}=\beta} \widehat{V}_{i_1} \hat{\otimes} \cdots \hat{\otimes} \widehat{V}_{i_p}$.

$U_q'(\mathfrak{g}) \curvearrowright \widehat{V}^{\otimes\beta} \curvearrowleft R(\beta)$ defined using R-matrices

$\rightsquigarrow \mathcal{F}_\beta \colon R(\beta)\text{-mod}^0 \to \mathcal{C}_\mathfrak{g}, \quad M \mapsto \widehat{V}^{\otimes\beta} \otimes_{R(\beta)} M$

$\mathcal{F} = \bigoplus_\beta \mathcal{F}_\beta \colon \bigoplus_\beta R(\beta)\text{-mod}^0 \to \mathcal{C}_\mathfrak{g}$: **gene'd Q-aff. SW duality functor**

Then we construct a $(U_q'(\mathfrak{g}), R(\beta))$-bimodule as follows.

V_i $(i \in J) \rightsquigarrow \widehat{V}_i = V_i[\![w]\!]$: a completed affinization $(U_q'(\mathfrak{g})$-module)

For $\beta \in Q^+$, $\widehat{V}^{\otimes\beta} = \bigoplus_{\alpha_{i_1}+\cdots+\alpha_{i_p}=\beta} \widehat{V}_{i_1} \hat{\otimes} \cdots \hat{\otimes} \widehat{V}_{i_p}$.

$U_q'(\mathfrak{g}) \curvearrowright \widehat{V}^{\otimes\beta} \curvearrowleft R(\beta)$ defined using R-matrices

$\rightsquigarrow \mathcal{F}_\beta \colon R(\beta)\text{-mod}^0 \to \mathcal{C}_\mathfrak{g}, \quad M \mapsto \widehat{V}^{\otimes\beta} \otimes_{R(\beta)} M$

$\mathcal{F} = \bigoplus_\beta \mathcal{F}_\beta \colon \bigoplus_\beta R(\beta)\text{-mod}^0 \to \mathcal{C}_\mathfrak{g}$: **gene'd Q-aff. SW duality functor**

Theorem ([KKK18])

(i) $\mathcal{F}$ is monoidal $(\mathcal{F}(M \circ M') \cong \mathcal{F}(M) \otimes \mathcal{F}(M')$, etc.).

(ii) If $\{R(\beta)\}$ are of type ADE, $\mathcal{F}$ is exact.

In [KKK15], the results of [KKK18] were applied in untwisted ADE types ($\mathfrak{g} = \widehat{\mathrm{g}}$), and gave a positive answer to the previous question (i.e., existence of a functor inducing $\bigoplus_\beta K(R(\beta)\text{-mod}^0)_\mathbb{C} \overset{\cong}{\to} K(\mathcal{C}_Q)_\mathbb{C}$)

In [KKK15], the results of [KKK18] were applied in untwisted ADE types ($\mathfrak{g} = \widehat{\mathrm{g}}$), and gave a positive answer to the previous question (i.e., existence of a functor inducing $\bigoplus_\beta K(R(\beta)\text{-mod}^0)_\mathbb{C} \overset{\cong}{\to} K(\mathcal{C}_Q)_\mathbb{C}$)

recall In the construction of $\mathcal{C}_Q$, a map $R^+ \ni \alpha \mapsto V^\alpha \in \mathcal{C}_{\hat{\mathrm{g}}}$ is used.

Take $\{V^{\alpha_i}\}_{i\in J}$ as the given data $\rightsquigarrow \mathcal{F} \colon \bigoplus_\beta R(\beta)\text{-mod}^0 \to \mathcal{C}_{\hat{\mathrm{g}}}$.

- In this case, $R(\beta)$ is of type g $\Rightarrow \mathcal{F}$ is exact.
- The image of $\mathcal{F} \colon \bigoplus_\beta R(\beta)\text{-mod}^0 \to \mathcal{C}_{\hat{\mathrm{g}}}$ is contained in $\mathcal{C}_Q$.

In [KKK15], the results of [KKK18] were applied in untwisted ADE types ($\mathfrak{g} = \widehat{\mathrm{g}}$), and gave a positive answer to the previous question

(i.e., existence of a functor inducing $\bigoplus_\beta K(R(\beta)\text{-mod}^0)_{\mathbb{C}} \overset{\cong}{\to} K(\mathcal{C}_Q)_{\mathbb{C}}$)

<u>recall</u> In the construction of $\mathcal{C}_Q$, a map $R^+ \ni \alpha \mapsto V^\alpha \in \mathcal{C}_{\hat{\mathrm{g}}}$ is used.

Take $\{V^{\alpha_i}\}_{i\in J}$ as the given data $\rightsquigarrow \mathcal{F}\colon \bigoplus_\beta R(\beta)\text{-mod}^0 \to \mathcal{C}_{\hat{\mathrm{g}}}$.

- In this case, $R(\beta)$ is of type g $\Rightarrow \mathcal{F}$ is exact.
- The image of $\mathcal{F}\colon \bigoplus_\beta R(\beta)\text{-mod}^0 \to \mathcal{C}_{\hat{\mathrm{g}}}$ is contained in $\mathcal{C}_Q$.

Theorem ([KKK15])

In this case, the gene'd QASW duality functor $\mathcal{F}\colon \bigoplus_\beta R(\beta)\text{-mod}^0 \to \mathcal{C}_Q$,

which is monoidal and exact, gives one-to-one corresp. between simples.

($\Rightarrow \bigoplus_\beta K(R(\beta)\text{-mod}^0) \overset{\sim}{\to} K(\mathcal{C}_Q)$)

$\mathcal{F}$: $\bigoplus_\beta R(\beta)\text{-mod}^0 \to \mathcal{C}_Q$ is monoidal, exact, gives one-to-one corresp. between simples.

Natrual problems

(i) Is this an equivalence?

(ii) Is there a generalization to the cases other than untwisted ADE types?

Both problems have been solved affirmatively!

$\mathcal{F}\colon \bigoplus_\beta R(\beta)\text{-mod}^0 \to \mathcal{C}_Q$ is monoidal, exact, gives one-to-one corresp. between simples.

Natrual problems

(i) Is this an equivalence?

(ii) Is there a generalization to the cases other than untwisted ADE types?

Both problems have been solved affirmatively!

Theorem ([Fujita, 17], [Fujita, 20])

The gene'd QASW duality functor $\mathcal{F}\colon \bigoplus_\beta R(\beta)\text{-mod}^0 \to \mathcal{C}_Q$ gives an equivalence of monoidal categories (in untwisted ADE types).

In the proof of [Fujita, 17], he used the geometric representation theory on quiver varieties and the theory of affine highest weight categories (we will return to this result later).

generalization to non-ADE cases

$\mathfrak{g}$: non-simply laced (untwisted or twisted) affine Lie algebra

Set a simple Lie algebra g to be as follows:

$U_q'(\mathfrak{g})$	$B_n^{(1)}$	$C_n^{(1)}$	$F_4^{(1)}$	$G_2^{(1)}$	$A_n^{(2)}$	$D_n^{(2)}$	$E_6^{(2)}$	$D_4^{(3)}$
g	A_{2n-1}	D_{n+1}	E_6	D_4	A_n	D_n	E_6	D_4

generalization to non-ADE cases

$\mathfrak{g}$: non-simply laced (untwisted or twisted) affine Lie algebra

Set a simple Lie algebra g to be as follows:

$U_q'(\mathfrak{g})$	$B_n^{(1)}$	$C_n^{(1)}$	$F_4^{(1)}$	$G_2^{(1)}$	$A_n^{(2)}$	$D_n^{(2)}$	$E_6^{(2)}$	$D_4^{(3)}$
g	A_{2n-1}	D_{n+1}	E_6	D_4	A_n	D_n	E_6	D_4

Similarly as ADE cases, define a map $R_{\mathrm{g}}^+ \ni \alpha \mapsto V^\alpha \in \mathcal{C}_{\mathfrak{g}}$,

and set $\mathcal{C}_Q = \langle V^\alpha \rangle_{\otimes,\text{ext.,subquot.}}$ (**Hernandez–Leclerc's subcategory**)

$\rightsquigarrow$ functor $\mathcal{F}$: $\bigoplus_\beta R(\beta)\text{-mod}^0 \to \mathcal{C}_Q$ ($\{R(\beta)\}$: quiver Hecke of type g)

Theorem ([KKK16], [Kashiwara–Oh, 19], [Oh–Scrimshaw, 19])

In all the above cases, the gene'd QASW duality functor $\mathcal{F}$ is monoidal, exact, and gives one-to-one correspondence between simple modules.

$$\left(\Rightarrow \bigoplus_\beta K(R(\beta)\text{-mod}^0) \xrightarrow{\sim} K(\mathcal{C}_Q).\right)$$

Summary

$U_q'(\mathfrak{g})$	monoidal	exact	bij. of simples	equiv.
ADE	◯	◯	◯	◯
others	◯	◯	◯	?

Summary

$U_q'(\mathfrak{g})$	monoidal	exact	bij. of simples	equiv.
ADE	◯	◯	◯	◯
others	◯	◯	◯	?

Theorem ([N])

In general types, the gene'd QASW duality functor $\mathcal{F}$ gives an equivalence of monoidal categories $\bigoplus_\beta R(\beta)\text{-mod}^0$ and $\mathcal{C}_Q$.

Proof to [Conjecture 5.7, KKK16], [Conjecture 6.11, KO19].

Corollary

Let $\mathfrak{g}^{(1)}$: untwisted ADE, $\mathfrak{g}^{(t)}$: twisted, ${}^L\mathfrak{g}^{(t)}$: the Langland dual of $\mathfrak{g}^{(t)}$

(e.g. $\mathfrak{g}^{(1)} = A^{(1)}_{2n-1}$, $\mathfrak{g}^{(2)} = A^{(2)}_{2n-1}$, ${}^L\mathfrak{g}^{(2)} = B^{(1)}_n$)

$\mathcal{C}_{Q^{(1)}}$, $\mathcal{C}_{Q^{(t)}}$, $\mathcal{C}_{{}^LQ}$: corresponding HL subcategories

Corollary

The monoidal categories $\mathcal{C}_{Q^{(1)}}$, $\mathcal{C}_{Q^{(t)}}$, $\mathcal{C}_{{}^LQ}$ are mutually equivalent.

$\because$ The corresponding quiver Hecke algebras $R(\beta)$ are the same. $\square$

Ex.

$$\begin{array}{ccccc} & & \mathcal{C}_{Q^{(1)}} \subseteq \mathcal{C}_{A^{(1)}_{2n-1}} & & \\ & & \uparrow\wr & & \\ \mathcal{C}_{A^{(2)}_{2n-1}} \supseteq \mathcal{C}_{Q^{(2)}} & \xleftarrow{\sim} & \bigoplus_\beta R^{A_{2n-1}}(\beta)\text{-mod}^0 & \xrightarrow{\sim} & \mathcal{C}_{{}^LQ} \subseteq \mathcal{C}_{B^{(1)}_n} \end{array}$$

strategy of the proof of $\mathcal{F}\colon \bigoplus R(\beta)\text{-mod}^0 \xrightarrow{\sim} \mathcal{C}_Q$

Fact $\mathcal{C}_Q$ has a block dec. $\mathcal{C}_Q = \bigoplus_\beta \mathcal{C}_{Q,\beta}$ $(\beta \in Q^+)$ such that

$\mathcal{F}_\beta\colon R(\beta)\text{-mod}^0 \to \mathcal{C}_{Q,\beta}$

$\therefore$ Enough to prove $\mathcal{F}_\beta\colon R(\beta)\text{-mod}^0 \xrightarrow{\sim} \mathcal{C}_{Q,\beta}$ for each β.

strategy of the proof of $\mathcal{F}\colon \bigoplus R(\beta)\text{-mod}^0 \xrightarrow{\sim} \mathcal{C}_Q$

<u>Fact</u> $\mathcal{C}_Q$ has a block dec. $\mathcal{C}_Q = \bigoplus_\beta \mathcal{C}_{Q,\beta}$ ($\beta \in Q^+$) such that
$\mathcal{F}_\beta\colon R(\beta)\text{-mod}^0 \to \mathcal{C}_{Q,\beta}$

$\therefore$ Enough to prove $\mathcal{F}_\beta\colon R(\beta)\text{-mod}^0 \xrightarrow{\sim} \mathcal{C}_{Q,\beta}$ for each β.

From the homological viewpoint, $R(\beta)\text{-mod}^0$ and $\mathcal{C}_{Q,\beta}$ are too small (e.g., not enough proj.)

$R(\beta) = \bigoplus_{n\in\mathbb{Z}} R(\beta)_n \rightsquigarrow \widehat{R}(\beta) = \prod_n R(\beta)_n$: completion (cf. $\mathbb{C}[z] \rightsquigarrow \mathbb{C}[\![z]\!]$), and consider $\widehat{R}(\beta)\text{-Mod}$ instead.

strategy of the proof of $\mathcal{F}\colon \bigoplus R(\beta)\text{-mod}^0 \xrightarrow{\sim} \mathcal{C}_Q$

Fact $\mathcal{C}_Q$ has a block dec. $\mathcal{C}_Q = \bigoplus_\beta \mathcal{C}_{Q,\beta}$ ($\beta \in Q^+$) such that
$\mathcal{F}_\beta\colon R(\beta)\text{-mod}^0 \to \mathcal{C}_{Q,\beta}$

$\therefore$ Enough to prove $\mathcal{F}_\beta\colon R(\beta)\text{-mod}^0 \xrightarrow{\sim} \mathcal{C}_{Q,\beta}$ for each β.

From the homological viewpoint, $R(\beta)\text{-mod}^0$ and $\mathcal{C}_{Q,\beta}$ are too small (e.g., not enough proj.)

$R(\beta) = \bigoplus_{n\in\mathbb{Z}} R(\beta)_n \rightsquigarrow \widehat{R}(\beta) = \prod_n R(\beta)_n$: completion (cf. $\mathbb{C}[z] \rightsquigarrow \mathbb{C}[\![z]\!]$), and consider $\widehat{R}(\beta)\text{-Mod}$ instead.

advantage $\circ$ $\widehat{R}(\beta)\text{-mod} = R(\beta)\text{-mod}^0$

strategy of the proof of $\mathcal{F}\colon \bigoplus R(\beta)\text{-mod}^0 \xrightarrow{\sim} \mathcal{C}_Q$

Fact $\mathcal{C}_Q$ has a block dec. $\mathcal{C}_Q = \bigoplus_\beta \mathcal{C}_{Q,\beta}$ $(\beta \in Q^+)$ such that

$\mathcal{F}_\beta\colon R(\beta)\text{-mod}^0 \to \mathcal{C}_{Q,\beta}$

$\therefore$ Enough to prove $\mathcal{F}_\beta\colon R(\beta)\text{-mod}^0 \xrightarrow{\sim} \mathcal{C}_{Q,\beta}$ for each β.

From the homological viewpoint, $R(\beta)\text{-mod}^0$ and $\mathcal{C}_{Q,\beta}$ are too small (e.g., not enough proj.)

$R(\beta) = \bigoplus_{n\in\mathbb{Z}} R(\beta)_n \rightsquigarrow \widehat{R}(\beta) = \prod_n R(\beta)_n$: completion (cf. $\mathbb{C}[z] \rightsquigarrow \mathbb{C}[\![z]\!]$), and consider $\widehat{R}(\beta)\text{-Mod}$ instead.

advantage
- $\widehat{R}(\beta)\text{-mod} = R(\beta)\text{-mod}^0$
- $\widehat{R}(\beta)\text{-Mod}$ is **affine highest weight category**!

(a generalization of highest weight cat. by Cline–Parshall–Scott.

$\Delta(\lambda)$: standard $\twoheadrightarrow L(\lambda)$: simple $\hookrightarrow \overline{\nabla}(\lambda)$: proper costandard)

$$\mathcal{F}_\beta \colon R(\beta)\text{-mod}^0 \to \mathcal{C}_{Q,\beta}, \qquad M \mapsto \widehat{V}^{\otimes\beta} \otimes_{R(\beta)} M$$

$(\widehat{V}^{\otimes\beta} \colon (U_q'(\mathfrak{g}), R(\beta))$-bimod.)

$$\mathcal{F}_\beta \colon \widehat{R}(\beta)\text{-mod} \to \mathcal{C}_{Q,\beta}, \qquad M \mapsto \widehat{V}^{\otimes\beta} \otimes_{R(\beta)} M$$

$$\widehat{R}(\beta)\text{-mod} \subset \widehat{R}(\beta)\text{-Mod}$$

(aff. h.w.)

($\widehat{V}^{\otimes\beta}\colon (U_q'(\mathfrak{g}), R(\beta))$-bimod.)

$$\mathcal{F}_\beta \colon \widehat{R}(\beta)\text{-mod} \to \mathcal{C}_{Q,\beta}, \qquad M \mapsto \widehat{V}^{\otimes\beta} \otimes_{\widehat{R}(\beta)} M$$

extend $\downarrow$ $\qquad \cap$

$$\mathcal{F}_\beta \colon \underset{\text{(aff. h.w.)}}{\widehat{R}(\beta)\text{-Mod}} \to \{U_q'(\mathfrak{g})\text{-modules}\} \quad (\widehat{V}^{\otimes\beta} \colon (U_q'(\mathfrak{g}), \widehat{R}(\beta))\text{-bimod.})$$

$$\mathcal{F}_\beta \colon \widehat{R}(\beta)\text{-mod} \to \mathcal{C}_{Q,\beta}, \qquad M \mapsto \widehat{V}^{\otimes\beta} \otimes_{\widehat{R}(\beta)} M$$

extend $\downarrow$ $\qquad \cap$

$$\mathcal{F}_\beta \colon \underset{\text{(aff. h.w.)}}{\widehat{R}(\beta)\text{-Mod}} \to \{U_q'(\mathfrak{g})\text{-modules}\} \quad (\widehat{V}^{\otimes\beta} \colon (U_q'(\mathfrak{g}), \widehat{R}(\beta))\text{-bimod.})$$

Theorem ([Fujita, 18])

A_i-Mod: affine h.w. $(i = 1, 2)$, $F \colon A_1\text{-Mod} \to A_2\text{-Mod}$: exact.

Assume (i) A_i is finitely generated over its center $(i = 1, 2)$,

(ii) $^\exists$bijection $f \colon \Pi_1 \to \Pi_2$ such that $F(\Delta(\pi)) = \Delta(f(\pi))$,

$F(\overline{\nabla}(\pi)) = \overline{\nabla}(f(\pi))$ for $^\forall\pi$.

Then F is an equivalence.

$$\mathcal{F}_\beta \colon \widehat{R}(\beta)\text{-mod} \to \mathcal{C}_{Q,\beta}, \qquad M \mapsto \widehat{V}^{\otimes\beta} \otimes_{\widehat{R}(\beta)} M$$

extend $\downarrow$ $\qquad\cap$

$$\mathcal{F}_\beta \colon \widehat{R}(\beta)\text{-Mod} \to \text{“}\widehat{\mathcal{C}}_{Q,\beta}\text{”}$$

(aff. h.w.) (aff. h.w.?)

($\widehat{V}^{\otimes\beta}$: $(U_q'(\mathfrak{g}), \widehat{R}(\beta))$-bimod.)

Theorem ([Fujita, 18])

A_i-Mod: affine h.w. $(i = 1, 2)$, $F \colon A_1\text{-Mod} \to A_2\text{-Mod}$: exact.

Assume (i) A_i is finitely generated over its center $(i = 1, 2)$,

(ii) $^{\exists}$bijection $f \colon \Pi_1 \to \Pi_2$ such that $F(\Delta(\pi)) = \Delta(f(\pi))$,

$F(\overline{\nabla}(\pi)) = \overline{\nabla}(f(\pi))$ for $^{\forall}\pi$.

Then F is an equivalence.

To prove $\mathcal{F}_\beta \colon R(\beta)\text{-mod}^0 \xrightarrow{\sim} \mathcal{C}_{Q,\beta}$, enough to show the following:

(i) Find an algebra A with an algebra homomorphism $\Phi \colon U_q'(\mathfrak{g}) \to A$.

(ii) Show that $\Phi^*|_{A\text{-mod}} \colon A\text{-mod} \to U_q'(\mathfrak{g})\text{-mod}$ gives an equivalence between $A\text{-mod}$ and $\mathcal{C}_{Q,\beta}$.

(iii) Define $\mathcal{F}'_\beta \colon \widehat{R}(\beta)\text{-Mod} \to A\text{-Mod}$ s.t. $\Phi^* \circ \mathcal{F}'_\beta|_{\widehat{R}(\beta)\text{-mod}} = \mathcal{F}_\beta$.

(iv) Show that $A\text{-Mod}$ is aff. h.w., and $\mathcal{F}'_\beta$ gives an equivalence $\mathcal{F}'_\beta \colon \widehat{R}(\beta)\text{-Mod} \xrightarrow{\sim} A\text{-Mod}$.

$$
\begin{array}{rccccc}
\mathcal{F}_\beta \colon & \widehat{R}(\beta)\text{-mod} & \to & A\text{-mod} & \overset{\Phi^*}{\xrightarrow{\sim}} & \mathcal{C}_{Q,\beta} \\
& \cap & & \cap & & \\
\mathcal{F}'_\beta \colon & \underset{\text{(aff. h.w.)}}{\widehat{R}(\beta)\text{-Mod}} & \to & A\text{-Mod} & &
\end{array}
$$

proof in untwisted ADE in [Fujita, 17]

(i) Find an algebra A with an algebra homomorphism $\Phi\colon U_q'(\mathfrak{g}) \to A$.

(ii) Show that $\Phi^*|_{A\text{-mod}}\colon A\text{-mod} \to U_q'(\mathfrak{g})\text{-mod}$ gives an equivalence between $A\text{-mod}$ and $\mathcal{C}_Q$.

(iii) Define $\mathcal{F}_\beta'\colon \widehat{R}(\beta)\text{-Mod} \to A\text{-Mod}$ s.t. $\Phi^* \circ \mathcal{F}_\beta'|_{\widehat{R}(\beta)\text{-mod}} = \mathcal{F}_\beta$.

(iv) Show that $A\text{-Mod}$ is aff. h.w., and $\mathcal{F}_\beta'$ gives an equivalence $\mathcal{F}_\beta'\colon \widehat{R}(\beta)\text{-Mod} \xrightarrow{\sim} A\text{-Mod}$.

In [Fujita, 17], he proved these statements with $A = \widehat{\mathcal{K}}^{\mathbb{G}}(Z^\bullet)$ (completed equiv. K-gps of the Steinberg type graded quiver var.)

(i) ${}^\exists\Phi\colon U_q'(\mathfrak{g}) \to \widehat{\mathcal{K}}^{\mathbb{G}}(Z^\bullet)$ by Nakajima,

(iii) define $\widehat{\mathcal{K}}^{\mathbb{G}}(Z^\bullet) \curvearrowright \widehat{V}^{\otimes\beta}$ geometrically,

(ii), (iv) work hard (omit)

proof in general types

(i) Find an algebra A with an algebra homomorphism $\Phi\colon U_q'(\mathfrak{g}) \to A$.

(ii) Show that $\Phi^*|_{A\text{-mod}}\colon A\text{-mod} \to U_q'(\mathfrak{g})\text{-mod}$ gives an equivalence between $A\text{-mod}$ and $\mathcal{C}_Q$.

(iii) Define $\mathcal{F}'_\beta\colon \widehat{R}(\beta)\text{-Mod} \to A\text{-Mod}$ s.t. $\Phi^* \circ \mathcal{F}'_\beta|_{\widehat{R}(\beta)\text{-mod}} = \mathcal{F}_\beta$.

(iv) Show that $A\text{-Mod}$ is aff. h.w., and $\mathcal{F}'_\beta$ gives an equivalence $\mathcal{F}'_\beta\colon \widehat{R}(\beta)\text{-Mod} \xrightarrow{\sim} A\text{-Mod}$.

There is no quiver var., and we adopt a completely different algebra A.

proof in general types

(i) Find an algebra A with an algebra homomorphism $\Phi\colon U_q'(\mathfrak{g}) \to A$.

(ii) Show that $\Phi^*|_{A\text{-mod}}\colon A\text{-mod} \to U_q'(\mathfrak{g})\text{-mod}$ gives an equivalence between $A\text{-mod}$ and $\mathcal{C}_Q$.

(iii) Define $\mathcal{F}'_\beta\colon \widehat{R}(\beta)\text{-Mod} \to A\text{-Mod}$ s.t. $\Phi^* \circ \mathcal{F}'_\beta|_{\widehat{R}(\beta)\text{-mod}} = \mathcal{F}_\beta$.

(iv) Show that A-Mod is aff. h.w., and $\mathcal{F}'_\beta$ gives an equivalence $\mathcal{F}'_\beta\colon \widehat{R}(\beta)\text{-Mod} \xrightarrow{\sim} A\text{-Mod}$.

There is no quiver var., and we adopt a completely different algebra A.

recall $\widehat{V}^{\otimes\beta}$: $(U_q'(\mathfrak{g}), \widehat{R}(\beta))$-bimod., $\mathcal{F}_\beta(M) := \widehat{V}^{\otimes\beta} \otimes_{\widehat{R}(\beta)} M$

Set $\mathbb{E}^\beta = \mathrm{End}_{\widehat{R}(\beta)^{\mathrm{opp}}}(\widehat{V}^{\otimes\beta})$ (**analog of Schur algebra**).

This $\mathbb{E}^\beta$ is our A. (i), (iii) are obvious.

Theorem ([N])

Set $\mathbb{E}^\beta = \mathrm{End}_{\widehat{R}(\beta)^{\mathrm{opp}}}(\widehat{V}^{\otimes\beta})$.

(i) The alg. hom. $\Phi\colon U_q'(\mathfrak{g}) \to \mathbb{E}^\beta$ induces an equiv. $\Phi^*\colon \mathbb{E}^\beta\text{-mod} \xrightarrow{\sim} \mathcal{C}_{Q,\beta}$.

(ii) $\mathbb{E}^\beta\text{-Mod}$ is aff. h.w., and $\mathcal{F}'_\beta$ gives an equiv. $\widehat{R}(\beta)\text{-Mod} \xrightarrow{\sim} \mathbb{E}^\beta\text{-Mod}$.

In the proof, the **affine cellular str.** of (a quotient of) $U_q'(\mathfrak{g})$ and $\mathbb{E}^\beta$ are used.

Theorem ([N])

Set $\mathbb{E}^{\beta} = \mathrm{End}_{\widehat{R}(\beta)^{\mathrm{opp}}}(\widehat{V}^{\otimes\beta})$.

(i) The alg. hom. $\Phi\colon U_q'(\mathfrak{g}) \to \mathbb{E}^{\beta}$ induces an equiv. $\Phi^*\colon \mathbb{E}^{\beta}\text{-mod} \xrightarrow{\sim} \mathcal{C}_{Q,\beta}$.

(ii) $\mathbb{E}^{\beta}$-Mod is aff. h.w., and $\mathcal{F}'_{\beta}$ gives an equiv. $\widehat{R}(\beta)\text{-Mod} \xrightarrow{\sim} \mathbb{E}^{\beta}\text{-Mod}$.

In the proof, the **affine cellular str.** of (a quotient of) $U_q'(\mathfrak{g})$ and $\mathbb{E}^{\beta}$ are used.

Thank you for your attention!