Demazure modules, Demazure crystals and the $X = M$ conjecture

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Plan of the talk

1. Relations between Demazure crystals and KR crystals.
   (i) Previous result by Schilling and Tingley.
   (ii) Main result.

2. Application: $X = M$ conjecture for $A_n^{(1)}$ and $D_n^{(1)}$.
   (i) What is the $X = M$ conjecture?
   (ii) The sketch of the proof.
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2. Application: $X = M$ conjecture for $A_n^{(1)}$ and $D_n^{(1)}$.
   (i) What is the $X = M$ conjecture?
   (ii) The sketch of the proof.
\( \mathfrak{g} \): affine Lie algebra, \( I = \{0, \ldots, n\}, I_0 = I \setminus \{0\} \),
\( \mathfrak{g}_0 \subseteq \mathfrak{g} \): simple Lie subalgebra corresponding to \( I_0 \),
\( W, W_0 \): Weyl groups, \( w_0 \in W_0 \): longest element,
\( P^+, P^+_0 \): sets of dominant integral weights,
\( U_q(\mathfrak{g}), U_q(\mathfrak{g}_0) \): quantized enveloping algebras,
\( U'_q(\mathfrak{g}) \subseteq U_q(\mathfrak{g}) \): quantum affine algebra without the degree operator,
\( \Lambda_i \in P^+ (i \in I) \): fundamental weights of \( \mathfrak{g} \),
\( \varpi_i \in P^+_0 (i \in I_0) \): fundamental weight of \( \mathfrak{g}_0 \).
**motivation**

\[ B(\Lambda) \]: crystal basis of the integrable highest weight \( U_q(g) \)-module with highest weight \( \Lambda \in P^+ \),

\( u_\Lambda \subseteq B(\Lambda) \): highest weight element.

**Theorem**

If finite \( U'_q(g) \)-crystal \( B \) is perfect (some technical condition), then we have an isomorphism of \( U'_q(g) \)-crystals

\[
B(\Lambda) \otimes B \cong B(\Lambda')
\]

for suitable \( \Lambda, \Lambda' \in P^+ \).

**Question:** What is the image of \( u_\Lambda \otimes B \) under the above isomorphism?

**Answer:** Demazure crystal (recalled below).
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*If finite \( U_q'(\mathfrak{g}) \)-crystal \( B \) is **perfect** (some technical condition), then we have an isomorphism of \( U_q'(\mathfrak{g}) \)-crystals*

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**B(Λ):** crystal basis of the integrable highest weight $U_q(g)$-module with highest weight $Λ \in P^+$,  

$u_Λ \subseteq B(Λ):$ highest weight element.

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*If finite $U'_q(g)$-crystal $B$ is perfect (some technical condition), then we have an isomorphism of $U'_q(g)$-crystals*

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**Question:** What is the image of $u_Λ \otimes B$ under the above isomorphism?

**Answer:** Demazure crystal (recalled below).
Kirillov-Reshetikhin crystal

$W^{r,\ell}$ ($r \in I_0, \ell \in \mathbb{Z}_{>0}$): Kirillov-Reshetikhin (KR) modules

: a class of irreducible finite-dimensional $U'_q(g)$-modules.

Theorem ([Okado, Schilling], [Fourier, Okado, Schilling])

(i) If $g$ is nonexceptional, $W^{r,\ell}$ has a crystal basis $B^{r,\ell}$ for each $r, \ell$ ($B^{r,\ell}$: KR crystal).

(ii) For each $r \in I_0, c_r \in \{1, 2, 3\}$ exists such that

$$B^{r,\ell} \text{ is perfect } \iff \ell \in \mathbb{Z}_{>0} c_r.$$  

Moreover if $g$ is simply-laced or twisted, then all $c_r$ are 1.

$\Rightarrow$ For any sequence $r_1, \ldots, r_p \in I_0$ and $\ell \in \mathbb{Z}_{>0},$

$B^{r_1, c_{r_1} \ell} \otimes \cdots \otimes B^{r_p, c_{r_p} \ell}$ is perfect.
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⇒ For any sequence $r_1, \ldots, r_p \in I_0$ and $\ell \in \mathbb{Z}_{>0}$,

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$$B_{r_1,c_{r_1}\ell} \otimes \cdots \otimes B_{r_p,c_{r_p}\ell} \text{ is perfect.}$$
For a crystal $B$, a subset $S \subseteq B$ and $i \in I$, we denote by $F_i(S)$ the subset

$$F_i(S) = \{ f_i^k(b) | b \in S, k \geq 0 \} \setminus \{0\} \subseteq B.$$ 

Let $w \in W$ with a reduced expression $w = s_i k \cdots s_i$. It is known that the subset

$$B_w(\Lambda) = F_i k \cdots F_i (u_\Lambda) \subseteq B(\Lambda)$$

does not depend on the choice of the choice of the expression.

**Definition (Kashiwara, ’93)**

$B_w(\Lambda)$ is called a **Demazure crystal**.
For a crystal $B$, a subset $S \subseteq B$ and $i \in I$, we denote by $F_i(S)$ the subset

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$B_w(\Lambda)$ is called a Demazure crystal.
For a subset \( S \) of a crystal, we denote its character by

\[
\text{ch} \ S = \sum_{b \in S} e^{\text{wt}(b)} \in \mathbb{Z}[P].
\]

**Theorem ([Kashiwara])**

\[
\text{ch} \ B_w(\Lambda) = D_w(e^{\Lambda}).
\]

If \( w = s_{i_k} \cdots s_{i_1} \) is a reduced expression, \( D_w \) is defined by \( D_w = D_{i_k} \cdots D_{i_1} \) where

\[
D_i(e^{\Lambda}) = \begin{cases} 
  e^{s_i(\Lambda)} + \cdots + e^{\Lambda} & \text{if } \langle \Lambda, \alpha_i^\vee \rangle \geq 0, \\
  0 & \text{if } \langle \Lambda, \alpha_i^\vee \rangle = -1, \\
  -e^{\Lambda+\alpha_i} - \cdots - e^{s_i(\Lambda)-\alpha_i} & \text{if } \langle \Lambda, \alpha_i^\vee \rangle \leq -2.
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Assume that \( \mathfrak{g} \) is nonexceptional. For given \( r_1, \ldots, r_p \in I_0 \) and \( \ell \in \mathbb{Z}_{>0} \), set

\[
B = B^{r_1,c_{r_1}\ell} \otimes \cdots \otimes B^{r_p,c_{r_p}\ell},
\]

and let \( i \in I \) and \( w \in W \) be elements satisfying

\[
w\Lambda_i = w_0(c_{r_1} \varpi_{r_1} + \cdots + c_{r_p} \varpi_{r_p}) + \Lambda_0.
\]

Then we have \( B(\ell\Lambda_0) \otimes B \xrightarrow{\sim} B(\ell\Lambda_i) \) as \( \mathcal{U}_q'(\mathfrak{g}) \)-crystals.

**Theorem (Schilling and Tingley, 2011)**

1. The image of \( u_{\ell\Lambda_0} \otimes B \) under the above isomorphism is \( B_w(\ell\Lambda_i) \).
2. The weight of the image of \( u_{\ell\Lambda_0} \otimes b \) is equal to \( \text{wt}(b) - \delta D(b) \), where \( D : B \rightarrow \mathbb{Z} \) is the energy function.
Assume that \( g \) is nonexceptional. For given \( r_1, \ldots, r_p \in I_0 \) and \( \ell \in \mathbb{Z}_{>0} \), set
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Let $\Psi : u_{\ell \Lambda_0} \otimes B \sim B_w(\ell \Lambda_i)$ be the isomorphism. Since $B(\ell \Lambda_i)$ is a $U_q(g)$-crystal, for each element $b \in B$ we have

$$\text{wt}(\Psi(u_{\ell \Lambda_0} \otimes b)) = \lambda + \ell \Lambda_0 + s\delta \in P$$

for some $\lambda \in P_0$ and $s \in \mathbb{Z}$ ($\delta$ is the null root).

On the other hand since $B(\ell \Lambda_0) \otimes B$ is a $U'_q(g)$-crystal, we have

$$\text{wt}(u_{\ell \Lambda_0} \otimes b) = \lambda + \ell \Lambda_0 \in P/\mathbb{Z}\delta.$$ 

The second statement says that a function $D : B \rightarrow \mathbb{Z}$ called energy function is defined, and it satisfies that

$$D(b) = -s.$$
The precise meaning of the second statement

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The second statement says that a function $D : B \rightarrow \mathbb{Z}$ called energy function is defined, and it satisfies that

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Proposition (combinatorial $R$-matrix)

For every KR crystals $B_1, B_2$, $\exists R : B_1 \otimes B_2 \rightarrow B_2 \otimes B_1$.

$H : B_1 \otimes B_2 \rightarrow \mathbb{Z}$ (local energy function)

- Constant on each $U_q(g_0)$-component,
- For $b_1 \otimes b_2 \in B_1 \otimes B_2$, $R(b_1 \otimes b_2) = \tilde{b}_2 \otimes \tilde{b}_1$,

\[
H(e_0(b_1 \otimes b_2)) = \begin{cases} 
H(b_1 \otimes b_2) + 1 & e_0(b_1 \otimes b_2) = e_0b_1 \otimes b_2, \\
H(b_1 \otimes b_2) - 1 & e_0(\tilde{b}_2 \otimes \tilde{b}_1) = e_0\tilde{b}_2 \otimes \tilde{b}_1, \\
H(b_1 \otimes b_2) & e_0(\tilde{b}_2 \otimes \tilde{b}_1) = \tilde{b}_2 \otimes e_0\tilde{b}_1, \\
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Definition of the energy function

**Proposition (combinatorial $R$-matrix)**

*For every KR crystals $B_1, B_2, !\exists R : B_1 \otimes B_2 \rightarrow B_2 \otimes B_1.*

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H(b_1 \otimes b_2) & e_0(b_1 \otimes b_2) = b_1 \otimes e_0b_2, \\
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\( D : B \to \mathbb{Z} \) (energy function)

\[
\text{def} \quad (1) \text{ In the case where } B = B^{r,s}:
\]

\[ D(b) := H(b^h \otimes b) \quad \text{for some special element } b^h \in B. \]

(2) In the case where \( B = B_1 \otimes \cdots \otimes B_p \):

For \( b_1 \otimes \cdots \otimes b_p \in B \) and \( 1 \leq i \leq j \leq p \), define \( b^{(i)}_j \in B_j \) by

\[
B_i \otimes B_{i+1} \otimes \cdots \otimes B_j \xrightarrow{\sim} B_j \otimes B_i \otimes \cdots \otimes B_{j-1}
\]

\[
b_i \otimes b_{i+1} \otimes \cdots \otimes b_j \mapsto b^{(i)}_j \otimes \tilde{b}_i \otimes \cdots \otimes \tilde{b}_{j-1}.
\]

Then \( D : B \to \mathbb{Z} \) is defined by

\[
D(b_1 \otimes \cdots \otimes b_p) := \sum_{1 \leq i \leq p} D(b^{(1)}_i) + \sum_{1 \leq i < j \leq p} H(b_i \otimes b^{(i+1)}_j).
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\( D : B \rightarrow \mathbb{Z} \) (energy function)

\[ \text{def} \]

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\[ b_i \otimes b_{i+1} \otimes \cdots \otimes b_j \mapsto b^{(i)}_j \otimes \tilde{b}_i \otimes \cdots \otimes \tilde{b}_{j-1}. \]

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\[ D : B \to \mathbb{Z} \text{ (energy function)} \]

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\[
D(b_1 \otimes \cdots \otimes b_p) := \sum_{1 \leq i \leq p} D(b_i^{(1)}) + \sum_{1 \leq i < j \leq p} H(b_i \otimes b_j^{(i+1)}).
\]
Rephrase the above theorem

Theorem

Set $B = B^{r_1,c_{r_1}} \otimes \cdots \otimes B^{r_p,c_{r_p}}$, and let $i \in I$ and $w \in W$ be elements such that

$$w(\Lambda_i) = w_0(c_{r_1} \varpi_{r_1} + \cdots + c_{r_p} \varpi_{r_p}) + \Lambda_0.$$

Then there exists an isomorphism of full subgraphs

$$\Psi : u_{\ell \Lambda_0} \otimes B \overset{\sim}{\rightarrow} B_w(\ell \Lambda_i)$$

which satisfies

$$\text{wt } \Psi(u_{\ell \Lambda_0} \otimes b) = \text{wt}(b) - \delta D(b) \quad \text{for } b \in B.$$
Set $B = B^{r_{1},c_{r_{1}}} \otimes \cdots \otimes B^{r_{p},c_{r_{p}}}$, and let $i \in I$ and $w \in W$ be elements such that

$$w(\Lambda_{i}) = w_{0}(c_{r_{1}} \varpi_{r_{1}} + \cdots + c_{r_{p}} \varpi_{r_{p}}) + \Lambda_{0}.$$ 

Then there exists an isomorphism of full subgraphs

$$\Psi : u_{\ell\Lambda_{0}} \otimes B \xrightarrow{\sim} B_{w}(\ell\Lambda_{i})$$

which satisfies

$$\text{wt} \Psi(u_{\ell\Lambda_{0}} \otimes b) = \text{wt}(b) - \delta D(b) \quad \text{for} \quad b \in B.$$
As a consequence of the above theorem, we obtain the following corollary:

\[ \sum_{b \in B} e^{\text{wt}(b) - \delta D(b)} = \text{ch } B_w(\ell \Lambda_i) \]
\[ = D_w(e^{\ell \Lambda_i}). \]

**Goal:** Generalize the above results to

\[ B = B^{r_1, c_{r_1}}_{\ell_1} \otimes \cdots \otimes B^{r_p, c_{r_p}}_{\ell_p} \]

for arbitrary \( \ell_1, \ldots, \ell_p \in \mathbb{Z}_{>0} \).

Since it is not perfect,

\[ B(\Lambda) \otimes B \not\cong B(\Lambda') \]

for any \( \Lambda, \Lambda' \in P^+ \).
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**Corollary**

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\sum_{b \in B} e^{\text{wt}(b) - \delta D(b)} = \text{ch} B_w(\ell \Lambda_i) = D_w(e^{\ell \Lambda_i}).
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Since it is **not perfect**, \(B(\Lambda) \otimes B \not\cong B(\Lambda')\)

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Assume that \( \mathfrak{g} \) is nonexceptional. For simplicity, we also assume that the tensor product

\[
B = B^{r_1,c_{r_1}}_{\ell_1} \otimes \cdots \otimes B^{r_p,c_{r_p}}_{\ell_p}
\]

satisfies \( \ell_1 \geq \cdots \geq \ell_p \). Define \( i_1, \ldots, i_p \in I \) and \( w_1, \ldots, w_p \in W \) by the elements satisfying

\[
w_1(\Lambda_{i_1}) = c_{r_1} w_0(\varpi_{r_1}) + \Lambda_0,
w_1w_2(\Lambda_{i_2}) = w_0(c_{r_1} \varpi_{r_1} + c_{r_2} \varpi_{r_2}) + \Lambda_0,
\]

\[
\vdots
\]

\[
w_1w_2 \cdots w_p(\Lambda_{i_p}) = w_0(c_{r_1} \varpi_{r_1} + \cdots + c_{r_p} \varpi_{r_p}) + \Lambda_0.
\]
Main theorem: a generalization of the above result

Assume that \( g \) is nonexceptional. For simplicity, we also assume that the tensor product

\[
B = B^{r_1,c_r}_{\ell_1} \otimes \cdots \otimes B^{r_p,c_r}_{\ell_p}
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\]
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w_1w_2 \cdots w_p(\Lambda_{i_p}) = w_0(c_{r_1} \sigma_{r_1} + \cdots + c_{r_p} \sigma_{r_p}) + \Lambda_0.
\]
Define a subset

\[ S \subseteq B((\ell_1 - \ell_2)\Lambda_{i_1}) \otimes B((\ell_2 - \ell_3)\Lambda_{i_2}) \otimes \cdots \otimes B(\ell_p\Lambda_{i_p}) \]

by

\[ S = F_{w_1}(u_{(\ell_1-\ell_2)\Lambda_{i_1}} \otimes F_{w_2}(u_{(\ell_2-\ell_3)\Lambda_{i_2}} \otimes \cdots \otimes F_{w_p}(u_{\ell_p\Lambda_{i_p}}) \cdots)) \].

**Theorem (N)**

There exists an isomorphism of full subgraphs

\[ \Psi : u_{\ell_1\Lambda_0} \otimes B^{r_1,c_{r_1}\ell_1} \otimes \cdots \otimes B^{r_p,c_{r_p}\ell_p} \sim S \]

which satisfies

\[ \text{wt} \; \Psi(u_{\ell_1\Lambda_0} \otimes b) = \text{wt}(b) - \delta D(b)\delta \quad \text{for} \; b \in B. \]
Define a subset

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Similarly as a Demazure crystal, the character of $S$ is calculated as follows:

**Lemma**

$$\text{ch } S = D_{w_1}(e^{(\ell_1-\ell_2)\Lambda_{i_1}} \cdot D_{w_2}(e^{(\ell_2-\ell_3)\Lambda_{i_2}} \cdots D_{w_p}(e^{\ell_p\Lambda_{i_p}})\cdots)).$$

Hence we have the following corollary:

**Corollary**

$$\sum_{b \in B} e^{\text{wt}(b) - \delta D(b)} = \text{ch } S = D_{w_1}(e^{(\ell_1-\ell_2)\Lambda_{i_1}} \cdot D_{w_2}(e^{(\ell_2-\ell_3)\Lambda_{i_2}} \cdots D_{w_p}(e^{\ell_p\Lambda_{i_p}})\cdots)).$$
Similarly as a Demazure crystal, the character of $S$ is calculated as follows:

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	ext{ch } S = D_{w_1} \left( e^{(\ell_1-\ell_2)\Lambda_{i_1}} \cdot D_{w_2} \left( e^{(\ell_2-\ell_3)\Lambda_{i_2}} \cdots D_{w_p} \left( e^{\ell_p \Lambda_{i_p}} \right) \right) \right).
$$

Hence we have the following corollary:

**Corollary**

$$
\sum_{b \in B} e^{\text{wt}(b)-\delta D(b)} = \text{ch } S
$$

$$
= D_{w_1} \left( e^{(\ell_1-\ell_2)\Lambda_{i_1}} \cdot D_{w_2} \left( e^{(\ell_2-\ell_3)\Lambda_{i_2}} \cdots D_{w_p} \left( e^{\ell_p \Lambda_{i_p}} \right) \right) \right).
$$
For a tensor product of (not necessarily perfect) KR crystals $B = B^{r_1, \ell_1} \otimes \cdots \otimes B^{r_p, \ell_p}$ and $\mu \in P^+_0$, we define

$$X(B, \mu, q) = \sum_{b \in B^\text{hw}_\mu} q^{D(b)} \quad (\text{1-dimensional sum}),$$

where $B^\text{hw}_\mu$ is a subset of $B$ defined by

$$B^\text{hw}_\mu = \{ b \in B \mid \tilde{e}_i(b) = 0 \text{ for } i \in I_0, \text{wt}(b) = \mu \}.$$

Conjecture (Hatayama, Kuniba, et al. '99)

For every $\mu \in P^+_0$, we have

$$X(B, \mu, q) = M(B, \mu, q),$$

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$$X(B, \mu, q) = M(B, \mu, q),$$

where $M(B, \mu, q) \in \mathbb{Z}[q]$ is the fermionic form defined below.
For simplicity, assume that $g$ is of type $A_n^{(1)}$, $D_n^{(1)}$ or $E_n^{(1)}$. The fermionic form $M(B, \mu, q)$ is defined as follows:

$$
M(B, \mu, q) = \sum_{m=\{m_u^{(i)} \in \mathbb{Z}_{\geq 0} \}_{i \in I_0, u \geq 1}} q^{c(m)} \prod_{i \in I_0, u \geq 1} \left[ p_u^{(i)} + m_u^{(i)} \right]_q,
$$

where

$$
c(m) = \frac{1}{2} \sum_{i, j \in I_0, u, v \geq 1} (\alpha_i, \alpha_j) \min\{u, v\} m_u^{(i)} m_v^{(j)} - \sum_{u \in \mathbb{Z}_{> 0}} \min\{\ell_j, u\} m_u^{(r_j)},
$$

$$
p_u^{(i)} = \sum_{j \in I_0; r_j = i} \min\{u, \ell_j\} - \sum_{j \in I_0, v \geq 1} (\alpha_i, \alpha_j) \min\{u, v\} m_j^{(v)}.
$$

($p_u^{(i)}$ is called the vacancy number).
Theorem

The $X = M$ conjecture has been proved in these cases:

- $\mathfrak{g} = A_n^{(1)}$, [Kirillov, Schilling, Shimozono, 2002],
- $\mathfrak{g}$: nonexceptional type, the rank of $\mathfrak{g}$ is sufficiently large,
  [Lecouvey, Okado, Shimozono, 2010] and
  [Okado, Sakamoto, 2010],
- $\forall \mathfrak{g}$, if $\ell_i = 1$ for all $i$ [N],
- Other special cases.

Using the results stated above, we can show the $X = M$ conjecture for type $A_n^{(1)}$ and $D_n^{(1)}$. 
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- Other special cases.

Using the results stated above, we can show the $X = M$ conjecture for type $A_n^{(1)}$ and $D_n^{(1)}$. 
For $\mu \in P_0^+$, let $V_0(\mu)$ denote the irreducible $g_0$-module.

In order to prove

$$X(B, \mu, q) = M(B, \mu, q)$$

for every $\mu \in P_0^+$, it suffices to show that

$$\sum_{\mu \in P_0^+} X(B, \mu, q) \text{ch} V_0(\mu) = \sum_{\mu \in P_0^+} M(B, \mu, q) \text{ch} V_0(\mu)$$

since $\text{ch} V_0(\mu)$ are linearly independent.
By definition, we have

\[ \sum_{\mu \in P_0^+} X(B, \mu, q) \mathrm{ch} \, V_0(\mu) = \sum_{b \in B} q^{D(b)} e^{\text{wt}(b)}. \]

Hence if \( g \) is nonexceptional, we have from the above corollary that

\[ \sum_{\mu \in P_0^+} X(B, \mu, q) \mathrm{ch} \, V_0(\mu) = D_{w_1} \left( e^{(\ell_1 - \ell_2) \Lambda_{i_1}} \cdot D_{w_2} \left( e^{(\ell_2 - \ell_3) \Lambda_{i_2}} \cdots D_{w_p} \left( e^{\ell_p \Lambda_{i_p}} \right) \cdots \right) \right), \]

where we set \( q = e^{-\delta} \).
By definition, we have

$$\sum_{\mu \in P_0^+} X(B, \mu, q) ch V_0(\mu) = \sum_{b \in B} q^D(b) e^{\text{wt}(b)}.$$ 

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where we set $q = e^{-\delta}$. 
On the other hand, the following theorem can be proved:

**Theorem (N)**

If $g$ is of type $A_n^{(1)}$, $D_n^{(1)}$ or $E_n^{(1)}$, then we have

$$
\sum_{\mu \in P_0^+} M(B, \mu, q) \text{ch } V_0(\mu) = D_{w_1} \left( e^{(\ell_1-\ell_2)\Lambda_{i_1}} \cdot D_{w_2} \left( e^{(\ell_2-\ell_3)\Lambda_{i_2}} \cdots D_{w_p} \left( e^{\ell_p\Lambda_{i_p}} \right) \cdots \right) \right),
$$

where we set $q = e^{-\delta}$. 
sketch of the proof.) Let $V(\Lambda)$ denote the irreducible highest weight $U_q(g)$-module. We define $S$ by the subspace of

$$V((\ell_1 - \ell_2)\Lambda_{i_1}) \otimes V((\ell_2 - \ell_3)\Lambda_{i_2}) \otimes \cdots \otimes V(\ell_p\Lambda_{i_p})$$

corresponding to the subset

$$S = F_{w_1}(u_{(\ell_1 - \ell_2)\Lambda_{i_1}} \otimes F_{w_2}(u_{(\ell_2 - \ell_3)\Lambda_{i_2}} \otimes \cdots \otimes F_{w_p}(u_{\ell_p\Lambda_{i_p}}) \cdots))$$

$$= \subseteq B((\ell_1 - \ell_2)\Lambda_{i_1}) \otimes B((\ell_2 - \ell_3)\Lambda_{i_2}) \otimes \cdots \otimes B(\ell_p\Lambda_{i_p}).$$

Then the classical limit of $S$ becomes a $g_0 \otimes \mathbb{C}[t]$-module.

By construction, we have

$$\text{ch } S = D_{w_1}(e^{(\ell_1 - \ell_2)\Lambda_{i_1}} \cdot D_{w_2}(e^{(\ell_2 - \ell_3)\Lambda_{i_2}} \cdots D_{w_p}(e^{\ell_p\Lambda_{i_p}}) \cdots)).$$
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On the other hand, it is proved by Di Francesco and Kedem that there exists a $\mathfrak{g}_0 \otimes \mathbb{C}[t]$-module $M$ such that

$$\text{ch } M = \sum_{\mu \in P^+_0} M(B, \mu, q) \text{ch } V_0(\mu).$$

Moreover, we can show that

$$S \cong M.$$

Hence we have

$$\sum_{\mu \in P^+_0} M(B, \mu, q) \text{ch } V_0(\mu) = \text{ch } M = \text{ch } S$$

$$= D_{w_1} \left( e^{(\ell_1-\ell_2)\Lambda_{i_1}} \cdot D_{w_2} \left( e^{(\ell_2-\ell_3)\Lambda_{i_2}} \cdots D_{w_p} \left( e^{\ell_p \Lambda_{i_p}} \cdots \right) \right) \right). \quad \Box$$
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As a consequence, we have:

**Corollary**

If $g$ is $A_n^{(1)}$ or $D_n^{(1)}$, then we have that

$$
\sum_{\mu \in P_0^+} X(B, \mu, q) \text{ch } V_0(\mu) = \sum_{\mu \in P_0^+} M(B, \mu, q) \text{ch } V_0(\mu).
$$

Hence the $X = M$ conjecture holds in this case.