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Minimum Feedback Node Sets in Trivalent Cayley Graphs

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SUMMARY A minimum feedback node set in a graph is a minimum node subset whose deletion makes the graph acyclic. Its detection in a dependency graph is important to recover from a deadlock configuration. A livelock configuration is also avoidable if a check point is set in each node in the minimum feedback node set. Hence, its detection is very important to establish dependable network systems. In this letter, we give a minimum feedback node set in a trivalent Cayley graph. Assuming that each word has n bits, for any node, we can judge if it is included in the set or not in constant time.
key words: *feedback node set, trivalent Cayley graphs, interconnection networks*

1. Introduction

Currently, studies of parallel and distributed computation are becoming more significant. Moreover, research on so-called massively parallel machines has been conducted enthusiastically in recent years. Hence, many complex topologies of interconnection networks have been proposed. Most graphs studied so far offer a high processor density while keeping their diameters small. However, the degrees of these famous topologies, such as hypercubes and star graphs, increase with graph sizes, that is, the number of nodes. Interconnection networks with variable degrees require a large number of I/O communication ports when they are applied to massively parallel machines.

To overcome this difficulty, some interconnection networks with constant degrees have been proposed [3], [4], [8], [9]. A trivalent Cayley graph [9] is one of these networks, which has constant degree 3. Much attention has been paid to this graph since it has regularity which cannot be found with famous topologies with constant degrees such as de Bruijn and Kautz graphs.

Finding feedback node sets in interconnection networks is one of the most important issues. Its detection in a dependency graph is important to recover from a deadlock configuration. A livelock configuration is also avoidable if a check point is set in each node in the feedback node set. For general graphs, it is proved that the problem of finding it belongs to the NP-complete class [2]. However, polynomial algorithms for finding minimum size feedback node sets in cocomparability graphs, convex bipartite graphs, interval graphs, and cyclically reducible graphs have been proposed [5], [6], [10]. In [7], a polynomial algorithm finding minimum feedback sets in graphs whose vertex degrees are at most 3 is proposed. These algorithms take polynomial time of the number of nodes. Hence, for graphs with many nodes, this approach is still impractical.

In this letter, we give a minimum feedback node set in a trivalent Cayley graph. Each node in the graph can be judged whether it is in the set or not in constant time.

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2. Preliminaries

This section gives a definition of a trivalent Cayley graph and a minimum feedback node set of a general graph. The lower bound of the cardinality of such set is also given.

Definition 1: [9] In an n-dimensional trivalent Cayley graph TC_n, each node has a label $\boldsymbol{a} = a_1a_2\cdots a_n$ which satisfies following two conditions:

- $a_i \in \{\pm 1, \pm 2, \cdots, \pm n\}$ $(1 \le i \le n)$,
- $|a_{i+1}| = (|a_i| \bmod n) + 1$ $(1 \le i \le n-1)$.

In the following, we denote $-a_i$ as $\underline{a_i}$. Each edge is of the type $(\boldsymbol{a}, \delta(\boldsymbol{a}))$ where $\delta \in \{g, f, f^{-1}\}$, defined as follows:

$$g(\boldsymbol{a}) = a_1a_2\cdots\underline{a_n},$$
$$f(\boldsymbol{a}) = a_2a_3\cdots a_n\underline{a_1},$$
$$f^{-1}(\boldsymbol{a}) = \underline{a_n}a_1a_2\cdots a_{n-1}.$$

Note that $\underline{\underline{a_i}} = a_i$.

TC_n is a symmetric undirected 3-regular graph. The numbers of nodes and edges of TC_n are $n2^n$ and $3n2^{n-1}$, respectively. Figure 1 shows an example of TC_3 which has 24 nodes and 36 edges. In this example, edges $(123, 12\underline{3})$, $(123, 23\underline{1})$ and $(123, \underline{3}12)$ are obtained by operations g, f (or equivalently f^{-1}) and f^{-1} (or equivalently f), respectively.

Definition 2: A feedback node set of a graph $G(V, E)$, where V and E represent its node and edge sets, respectively, is a subset of nodes $S \subseteq V$ whose deletion from G induces an acyclic graph $G'(V', E')$ with $V' = V \backslash S$ and $E' = \{(\boldsymbol{u}, \boldsymbol{v}) \in E; \boldsymbol{u}, \boldsymbol{v} \in V'\}$. If

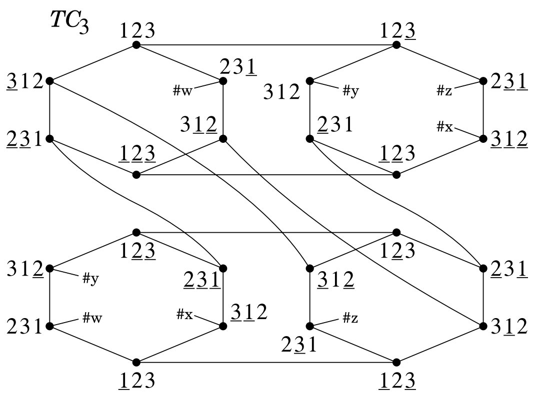

Fig. 1 TC_3.

the cardinality of S is the minimum possible, we call it a minimum feedback node set of G.

The lower bound of the cardinality of a minimum feedback node set for a general graph is given as follows.

Lemma 1: [1] Any feedback node set in a graph $G(V, E)$ with maximum degree r has at least $(|E| - |V| + 1)/(r - 1)$ nodes.

Corollary 1: A minimum feedback node set of the TC_n has at least $n2^{n-2} + 1$ nodes.

3. Minimum Feedback Node Set in TC_n

In $TC_n(V, E)$, consider a subset $S \subset V$ that consists of node $12 \cdots n$ and all nodes whose first and last elements are both negative, i.e.,

$$S = \{a_1 a_2 \cdots a_n \mid a_1 < 0, a_n < 0\} \cup \{12 \cdots n\}.$$

Note that $|S| = n2^{n-2} + 1$. If S is a feedback node set of TC_n, it is one of minimum feedback node sets.

In the following, we prove that S is a minimum feedback node set in TC_n. Let TC'_n be the subgraph of TC_n induced by $V \backslash S$.

We classify each node $\boldsymbol{a} = a_1 a_2 \cdots a_n$ in TC'_n into following three subsets according to its first and last elements:

$$S_{+,+} = \{\boldsymbol{a} \mid a_1 > 0, a_n > 0\}$$
$$S_{-,+} = \{\boldsymbol{a} \mid a_1 < 0, a_n > 0\}$$
$$S_{+,-} = \{\boldsymbol{a} \mid a_1 > 0, a_n < 0\}$$

Note that node $12 \cdots n$ is not included in $S_{+,+}$. Figure 2 is the state transition diagram where a state represents one of the subsets of nodes and a transition is denoted by a directed edge and its corresponding operations.

Lemma 2: Any node in $S_{-,+}$ is not included in a cycle in TC'_n.

Proof: Assume a cycle that includes node in $S_{-,+}$.

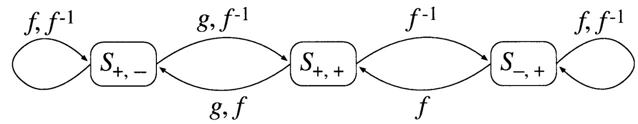

Fig. 2 State transition diagram.

Case 1 (All nodes in the cycle are in $S_{-,+}$.) The cycle must be obtained by repeating operation f (or equivalently f^{-1}) $2n$ times. However, any such cycle must contain a node whose first element is negative. Hence, there is no cycle whose nodes are all in $S_{-,+}$.

Case 2 (The cycle includes a node which is not in $S_{-,+}$.) The cycle has a subpath beginning with an edge from $S_{+,+}$ to $S_{-,+}$ obtained by operation f^{-1} and ending with an edge from $S_{-,+}$ to $S_{+,+}$ obtained by operation f. Since operation g cannot be applied to $S_{-,+}$, this subpath includes a path $\boldsymbol{u} \rightarrow \boldsymbol{v} \rightarrow \boldsymbol{w}$ of length 2 where edges $(\boldsymbol{u}, \boldsymbol{v})$ and $(\boldsymbol{v}, \boldsymbol{w})$ are obtained by operations f^{-1} and f, respectively. However, from the definition, these two operations cannot be applied in succession.

From above discussion, there is no cycle that includes a node in $S_{-,+}$. □

Lemma 3: There is no cycle in TC'_n which consists of nodes in $S_{+,-}$ only.

Proof: It is trivial from case 1 in the proof of Lemma 2. □

Lemma 4: Any edge $(\boldsymbol{u}, g(\boldsymbol{u}))$ where $\boldsymbol{u} \in S_{+,-}$ is not included in a cycle in TC'_n.

Proof: Assume that a cycle includes edge $(\boldsymbol{u}, g(\boldsymbol{u}))$ where $\boldsymbol{u} \in S_{+,-}$. Then, from Fig. 2, the cycle must be obtained by an alternative repetition of operations g and f^+ where f^+ represents one or more iterations of operation f.

Case 1 (The cycle includes a node that has more than one negative elements.) By an alternative repetition of operations g and f^+, the preceding negative element in the node is moved to left and finally it appears at the first position which is a contradiction. Hence, there is no cycle in this case.

Case 2 (Any node in the cycle has at most one negative element.) In such a cycle, there must be node $12 \cdots n$ which is deleted already. Hence, there is no such cycle. □

Lemma 5: Any edge $(\boldsymbol{u}, f^{-1}(\boldsymbol{u}))$ where $\boldsymbol{u} \in S_{+,-}$ is not included in a cycle in TC'_n.

Proof: Assume that a cycle in TC'_n includes edge $(\boldsymbol{u}, f^{-1}(\boldsymbol{u}))$ where $\boldsymbol{u} \in S_{+,-}$. Then from Lemma 4 the cycle must be obtained by an alternative repetition of operations g and $(f^{-1})^+$ where $(f^{-1})^+$ represents one or more iterations of operation f^{-1}. Then from the similar discussion to Lemma 4, Lemma 5 can be proved. □

From lemmas above, we obtain the followings.

Lemma 6: TC'_n does not have a cycle.

Theorem 1: S is a minimum feedback node set of TC_n.

By similar discussions, we obtain the following.

Corollary 2: In $TC_n(V, E)$, consider subset $S' \subset V$ that consists of node $\underline{1}\,\underline{2} \cdots \underline{n}$ and all nodes whose first and last elements are both positive. Then, S' is a minimum feedback node set of TC_n.

4. Conclusion

In this letter, we have presented a minimum feedback node set in an n-dimensional trivalent Cayley graph. The set consists of node $12 \cdots n$ and all nodes whose first and last elements are both negative. Assuming that each word has n bits, for any node, we can judge if it is included in the set or not in constant time. Future works include developments of the effective algorithms for finding minimum feedback node sets in other famous topologies.

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