ON THE DIJKGRAAF-WITTEN INVARIANT AND THE QUANDLE COCYCLE INVARIANT

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Abstract. We present a relation between the Dijkgraaf-Witten invariant and the quandle cocycle invariant. The quandle cocycle invariant of a twist-spun knot associated with a cyclic group of odd order is related to the Dijkgraaf-Witten invariant of the 2-fold branched covering manifold of the 3-sphere branched along the knot. We use covering presentations of 3-manifolds to show it.

1. Introduction

For a closed oriented 3-manifold M, the Dijkgraaf-Witten invariant [4] is given by the state sum,

$$Z_\theta(M) = \frac{1}{|G|} \sum_{\gamma \in \mathrm{Hom}(\pi_1(M), G)} \langle \gamma^*[\theta], [M] \rangle.$$

Here $[\theta]$ is the cohomology class of $H^3(BG, U(1))$, and $[M]$ is the fundamental class of M. γ^* is the map $H^3(BG, U(1)) \to H^3(M, U(1))$ induced by the classifying map $M \to BG$ corresponding to a representation $\gamma : \pi_1(M) \to G$. Wakui [10] gave a formulation of the Dijkgraaf-Witten invariant using triangulations, and proved its topological invariance in a rigorous way, which depends only on a group and the cohomology class of its 3-cocycle. Further he showed that the formulation can be extended for 3-manifolds with boundaries, and the construction gives an example of the topological quantum field theory. In [5], the author reconstructed the Dijkgraaf-Witten invariant using covering presentations of 3-manifolds. By a covering presentation of a closed oriented 3-manifold, we mean a link diagram, of the branch set in the base space of a simple branched covering from the 3-manifold to the 3-sphere S^3, together with information of the covering.

The quandle cocycle invariant [2] is also a state sum invariant, for oriented knots and surface-knots. They are defined with a quandle and its 2- or 3-cocycle, respectively. The shadow cocycle invariant [3] is defined for links with quandle 3-cocycles, as an application to the quandle cocycle invariants. Extended the definition to tangles, this invariant has been used for the calculations of the quandle cocycle invariant of the twist-spun knots ([8], [1] and [6], for example).

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This paper presents relations between the Dijkgraaf-Witten invariant, the shadow cocycle invariant and the quandle cocycle invariant. We use Abelian groups for the Dijkgraaf-Witten invariant, and their core racks for cocycle invariants. Quandle 3-cocycles which we deal with are particular ones derived from group 3-cocycles. Under these restrictions, the shadow cocycle invariant of a link is expressed by a 3-manifold invariant. This 3-manifold is the 2-fold branched covering space of S^3 branched along the link. In particular, if the group is the cyclic group of odd order, it turns out to be just the Dijkgraaf-Witten invariant up to constants. Furthermore, we show that the quandle cocycle invariant of a twist-spun knot can be computed using the shadow cocycle invariant of the knot in this case. Therefore the Dijkgraaf-Witten invariant is related to the quandle cocycle invariant.

This paper is organized as follows. Next section is devoted to the review of the Dijkgraaf-Witten invariant defined on covering presentations, associated with Abelian groups. The relation between the Dijkgraaf-Witten invariant and the shadow cocycle invariant is given in Section 3, and the relation between the quandle and the shadow cocycle invariants will be stated in Section 4.

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2. The Dijkgraaf-Witten invariant

We review the Dijkgraaf-Witten invariant of closed oriented 3-manifolds defined on their covering presentations, associated with Abelian groups. Refer to [5] for the detailed argument.

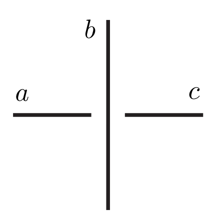
Let L be an unoriented link, and D_L its diagram. The symbol G denotes an Abelian group written additively. A *group coloring* on D_L in G is defined to be a map

$$C : \{\text{arcs of } D_L\} \to G$$

such that at each crossing,

$$C(a) = 2C(b) - C(a),$$

where a and c are the under-arcs and b is the over-arc as illustrated in Figure 1.



$$C(c) = 2C(b) - C(a)$$

Figure 1. The condition of group coloring

It is easy to verify that the number of colorings on D_L in G is an invariant of the link L. Furthermore, we have the identity that

$$\sharp\{\text{colorings on } D_L \text{ in } G\} = |G| \cdot \sharp\{\text{representations } \pi_1(M_2(L)) \to G\},$$

where $M_2(L)$ is the 2-fold branched covering space of S^3 branched along $L \subset S^3$. Hence it is a topological invariant of the 3-manifold. See the proof of Proposition 3.2 and 3.3 in [5], putting the label $\langle 12 \rangle$ on each arc of D_L and adding two trivial knots labeled $\langle 23 \rangle$ and $\langle 34 \rangle$.

A *region coloring* of $\mathbb{R}^2 \setminus D_L$, with respect to a coloring C on D_L in G, will be a map

$$S : \{\text{regions of } \mathbb{R}^2 \setminus D\} \to G$$

such that two colors on the adjacent regions separated by an arc a are expressed as s and $C(a) - s$ as shown in Figure 2. Since it is well defined around any crossing, a region coloring is uniquely determined by the color on the unbounded region.

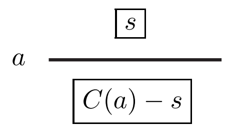
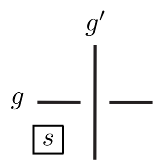


FIGURE 2. The condition of region coloring

A map $\theta : G \times G \times G \to A$, where A is another Abelian group written multiplicatively, is a *group 3-cocycle* if, by definition, it satisfies the identity

(GC1) $\theta(y, z, w) \cdot \theta(x + y, z, w)^{-1} \cdot \theta(x, y + z, w) \cdot \theta(x, y, z + w)^{-1} \cdot \theta(x, y, z) = 1_A$

for any $x, y, z, w \in G$. We define the *weight* $X_\theta(x; C, S)$ at a crossing x of a diagram D_L, with a group coloring C and a region coloring S, associated with a group 3-cocycle θ by

$X_\theta($ 

$) = \theta(g, -g + g', -s + g - g') \cdot \theta(g', g - g', s - g)$

$$\cdot \theta(-g + 2g', g - g', -s) \cdot \theta(g', -g + g', s - g').$$

Here s is the color on a region with the under-arc left towards the crossing, and g, g' are the colors on the under- and over-arcs touching the region, respectively.

If a group 3-cocycle θ satisfies the conditions,

(GC2) $$\theta(0, x, y) = \theta(x, 0, y) = \theta(x, y, 0) = 1_A$$

and

(GC3) $$\theta(x, -x, y) = \theta(x, y, -y) = 1_A,$$

then the weight $X_\theta(x; C, S)$ does not depend on the choice of two regions around x, and the expression

$$I_\theta(L) = \sum_C \sum_S \prod_x X_\theta(x; C, S) \in \mathbb{Z}[A]$$

gives an invariant of the link L and the 3-manifold $M_2(L)$. Here we take the product for all the crossings of D_L, the inner sum for all the region coloring, and the outer sum for all the group coloring on D_L. A large number of cohomology classes are realized by group 3-cocycles having the properties (GC2) and (GC3), though we do not have a nontrivial 3-cocycle with these properties in $\mathbb{Z}_2$ and $\mathbb{Z}_3$. Furthermore, the identity

$$I_\theta(L) = |G|^3 \cdot Z_\theta(M_2(L))$$

holds for the Dijkgraaf-Witten invariant $Z_\theta(M_2(L))$. Each contribution is presented by

$$\prod_{x \text{ of } D} X_\theta(x, C, S) = \langle \gamma^*[\theta], [M_2(L)] \rangle$$

for a representation $\gamma : \pi_1(M_2(L)) \to G$ corresponding to C. It shows that the contribution does not depend on the region colorings.

3. Shadow cocycle invariants

We first introduce the shadow cocycle invariant of links, and then show that in particular cases, it gives a 3-maifold invariant related to the Dijkgraaf-Witten invariant.

Let Q be a quandle. A *quandle coloring* on a diagram D_L of an oriented link L in Q is defined to be a map

$$C : \{\text{arcs of } D_L\} \to Q$$

such that

$$C(c) = C(a) * C(b)$$

at each crossing, where b is the over-arc, a is its left, and c is its right under-arcs, with respect to the orientation as illustrated in Figure 3. The symbol $*$ is the operation in Q.

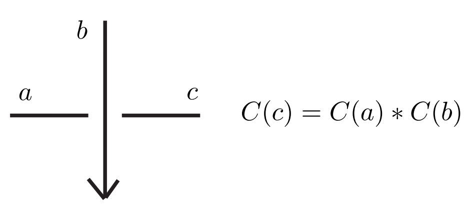


Figure 3. The condition of quandle coloring

A *shadow coloring* of $\mathbb{R}^2 \setminus D$, with respect to a quandle coloring C on D_L in Q, is a map

$$S : \{\text{regions of } \mathbb{R}^2 \setminus D\} \to Q$$

such that the two colors on the right and left regions of an arc a are expressed as s and $s * C(a)$ respectively, as shown in Figure 2. It is determined uniquely by the color on the unbounded region.

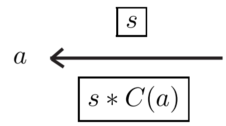


FIGURE 4. The condition of shadow coloring

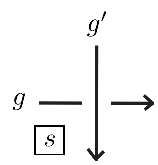
Let $\phi : Q \times Q \times Q \to A$ be a quandle 3-cocycle of Q in an Abelian group A, that is, a map satisfying the following conditions;

(QC1) $\phi(x, y, z) \cdot \phi(x * z, y * z, w) \cdot \phi(x, z, w)$
$= \phi(x * y, z, w) \cdot \phi(x, y, w) \cdot \phi(x * w, y * w, z * w)$,

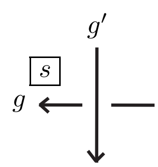
(QC2) $\phi(x, y, y) = 1_A$ and

(QC3) $\phi(x, x, y) = 1_A$,

for any $x, y, z, w \in Q$. Using a quandle 3-cocycle ϕ, we define the *weight* $W_\phi(x; C, S)$ at a crossing x of a diagram D_L, with a quandle coloring C and a shadow coloring S in Q as follows in two types of crossings.

$W_\phi($  $) = \phi(s, g, g')$,

and

$W_\phi($  $) = \phi(s, g, g')^{-1}$.

The *shadow cocycle invariant* [3] is the state-sum

$$\Psi_\phi(L) = \sum_C \sum_S \prod_x W_\phi(x; C, S) \in \mathbb{Z}[A],$$

which is an invariant of the link L. We remark that the condition (QC3) for quandle 3-cocycles is not needed to show the invariance.

Let us consider the case that the quandle Q_G is the core rack of an Abelian group G, with the operation

$$g * g' = 2g' - g$$

for any elements $g,\ g' \in G$. For a group 3-cocycle $\theta : G \times G \times G \to A$, we define a map

$$\tilde{\theta} : Q_G \times Q_G \times Q_G \to A$$

by

$$\begin{aligned}(x,y,z) \ \mapsto\ & \theta(2y,-2y+2z,-x+2y-2z)\cdot\theta(2z,2y-2z,x-2y)\\ &\cdot\theta(-2y+4z,2y-2z,-x)\cdot\theta(2z,-2y+2z,x-2z).\end{aligned}$$

Lemma 3.1. *If a group 3-cocycle θ of an Abelian group G in another Abelian group A satisfies the conditions (GC2) and (GC3), then the map $\tilde{\theta}$ given by θ as above is an quandle 3-cocycle.*

Proof. We show that the map $\tilde{\theta}$ satisfies the conditions (QC1), (QC2) and (QC3).

(QC1) $(\mathrm{LHS})\cdot(\mathrm{RHS})^{-1}$

$$\begin{aligned}
&= \tilde{\theta}(x,y,z)\cdot\tilde{\theta}(x*z,y*z,w)\cdot\tilde{\theta}(x,z,w)\\
&\quad\cdot\tilde{\theta}(x*y,z,w)^{-1}\cdot\tilde{\theta}(x,y,w)^{-1}\cdot\tilde{\theta}(x*w,y*w,z*w)^{-1}\\
&= \tilde{\theta}(x,y,z)\cdot\tilde{\theta}(2z-x,2z-y,w)\cdot\tilde{\theta}(x,z,w)\\
&\quad\cdot\tilde{\theta}(2y-x,z,w)^{-1}\cdot\tilde{\theta}(x,y,w)^{-1}\cdot\tilde{\theta}(2w-x,2w-y,2w-z)^{-1}\\
&= \theta(2y,-2y+2z,-x+2y-2z)\cdot\theta(2z,2y-2z,x-2y)\\
&\quad\cdot\theta(-2y+4z,2y-2z,-x)\cdot\theta(2z,-2y+2z,x-2z)\\
&\quad\cdot\theta(4z-2y,-4z+2y+2w,x-2y+2z-2w)\cdot\theta(2w,4z-2y-2w,-x+2y-2z)\\
&\quad\cdot\theta(-4z+2y+4w,4z-2y-2w,-2z+x)\cdot\theta(2w,-4z+2y+2w,2z-x-2w)\\
&\quad\cdot\theta(2z,-2z+2w,-x+2z-2w)\cdot\theta(2w,2z-2w,x-2z)\\
&\quad\cdot\theta(-2z+4w,2z-2w,-x)\cdot\theta(2w,-2z+2w,x-2w)\\
&\quad\cdot\theta(2z,-2z+2w,-2y+x+2z-2w)^{-1}\cdot\theta(2w,2z-2w,2y-x-2z)^{-1}\\
&\quad\cdot\theta(-2z+4w,2z-2w,-2y+x)^{-1}\cdot\theta(2w,-2z+2w,2y-x-2w)^{-1}\\
&\quad\cdot\theta(2y,-2y+2w,-x+2y-2w)^{-1}\cdot\theta(2w,2y-2w,x-2y)^{-1}\\
&\quad\cdot\theta(-2y+4w,2y-2w,-x)^{-1}\cdot\theta(2w,-2y+2w,x-2w)^{-1}\\
&\quad\cdot\theta(4w-2y,2y-2z,-2w+x-2y+2z)^{-1}\cdot\theta(4w-2z,-2y+2z,-x-2w+2y)^{-1}\\
&\quad\cdot\theta(2y+4w-4z,-2y+2z,-2w+x)^{-1}\cdot\theta(4w-2z,2y-2z,-x-2w+2z)^{-1}\\
&= \theta(-2y+2w,2z-2w,-x+2y-2z)^{-1}\cdot\theta(2y,-2y+2w,2z-2w)^{-1}\\
&\quad\cdot\theta(2z-2w,2y-2z,x-2y)\cdot\theta(2w,2z-2w,2y-2z)\\
&\quad\cdot\theta(-2y+2z,2y-2w,-x)^{-1}\cdot\theta(-2z+4w,-2y+2z,2y-2w)^{-1}\\
&\quad\cdot\theta(-2y+4z-2w,-2z+2w,x-2w)\cdot\theta(2y-4z+4w,-2y+4z-2w,-2z+2w)\\
&\quad\cdot\theta(-2z+2w,2y-2z,-x+2z-2w)^{-1}\cdot\theta(2w,-2z+2w,2y-2z)^{-1}\\
&\quad\cdot\theta(-2y+2z,2y-4z+2w,x-2y+2z-2w)\cdot\theta(2z,-2y+2z,2y-4z+2w)\\
&\quad\cdot\theta(x-2z+2w,4z-2y-2w,2y-2z)^{-1}\cdot\theta(2w,4z-2y-2w,2y-2z)\\
&\quad\cdot\theta(-x+2y,-2y+2w,2y-2z)^{-1}\cdot\theta(2w,-2y+2w,2y-2z)\\
&= \theta(-2y+2z,-2z+2w,2y-2z)\cdot\theta(2y-2z,-2y+2w,2y-2z)\\
&= 1_A.
\end{aligned}$$

The conditions (GC1), (GC2) and (GC3) of θ are used repeatedly.

(QC2) $\tilde{\theta}(x,y,y)=\theta(y,0,-x)\cdot\theta(y,0,x-y)\cdot\theta(y,0,-x)\cdot\theta(y,0,x-y)=1_A.$

(QC3) $\tilde{\theta}(x,x,y) = \theta(2x,-2x+2y,x-2y)\cdot\theta(2y,2x-2y,-x)$

$\cdot\,\theta(-2x+4y,2x-2y,-x)\cdot\theta(2y,-2x+2y,x-2y)$

$= 1_A\cdot 1_A$

$= 1_A,$

since

$$\theta(x+y,-y,w)\cdot\theta(x,y,-y+w) = 1_A$$

by putting z to be $-y$ in (GC1).

□

Theorem 3.2. *Let G be an Abelian group, and $\tilde{\theta}$ be the quandle 3-cocycle given by a group 3-cocycle θ of G satisfying the conditions (GC2) and (GC3). On the shadow cocycle invariant $\Psi_{\tilde{\theta}}(L)$ associated with $\tilde{\theta}$, we have the identity that*

$$\Psi_{\tilde{\theta}}(L) = |G|\sum_{\gamma\in Hom(\pi_1(M_2(L)),G)}\langle(2\gamma)^*[\theta],[M_2(L)]\rangle.$$

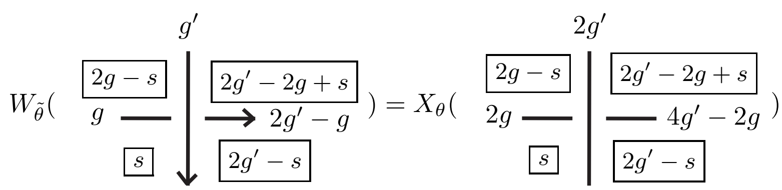
Here $M_2(L)$ is the 2-fold branched covering space of S^3 branched along the link L, and $2\gamma:\pi_1(M_2(L))\to G$ is the mapping $l\mapsto 2\gamma(l)$ for each loop l in $\pi_1(M_2(L))$.

In particular, in the case that $G=\mathbb{Z}_n$ of odd order n,

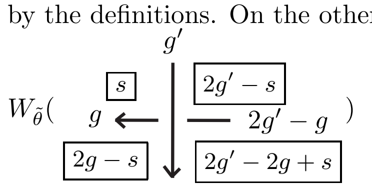
$$\Psi_{\tilde{\theta}}(L) = |G|^2\cdot Z_\theta(M_2(L)),$$

where Z_θ is the Dijkgraaf-Witten invariant.

Proof. Any quandle coloring C in Q_G can be seen as a group coloring in G with the same mapping. For the weight $W_{\tilde{\theta}}(x;C,S)$ at a crossing x of D, with a quandle coloring C and a shadow coloring S in Q, the following equalities hold for the two types of crossing.



by the definitions. On the other hand,



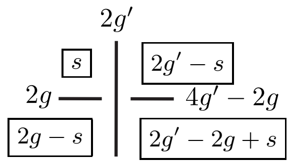
$= \tilde{\theta}(s,g,g')^{-1}$

$= \underline{\theta(2g,-2g+2g',-s+2g-2g')^{-1}}\cdot\underline{\underline{\theta(2g',2g-2g',s-2g)^{-1}}}$

$\cdot\,\theta(-2g+4g',2g-2g',-s)^{-1}\cdot\theta(2g',-2g+2g',s-2g')^{-1}$

$= \underline{\theta(2g,-2g+2g',s-2g')}\cdot\underline{\underline{\theta(2g,-2g+2g',s-2g')}}$

$\cdot\,\theta(2g',-2g+2g',-s+2g+2g')\cdot\theta(-2g+4g',2g-2g',s-2g)$

$= X_\theta($  $).$

Therefore, taking the product of the weights $W_{\tilde{\theta}}(x, C, S)$ for all the crossings of D,

$$\prod_{x \text{ of } D} W_{\tilde{\theta}}(x, C, S) = \prod_{x \text{ of } D} X_\theta(x, 2C, S).$$

Here the region coloring S in the right hand side is just the same mapping as the shadow coloring S in the left hand side. The coloring $2C$ represent the mapping

$$a \mapsto 2C(a) \in G$$

on each arc a of D. Recall that a group coloring corresponds to a representation $\pi_1(M) \to G$. Let γ be the representation corresponding to C. Then the representation corresponding to $2C$ is 2γ. Hence

$$\prod_{x \text{ of } D} X_\theta(x, 2C, S) = \langle 2\gamma^*[\theta], [M] \rangle$$

for each pair of a group coloring C and a region coloring S, and we obtain the identities in the statement. □

We see that the weight $W_{\tilde{\theta}}(x, C, S)$ does not depend on the orientations of the arcs by the proof. Iwakiri [6] showed it for the dihedral quandle R_p, that is the core rack of $\mathbb{Z}_p$ of odd prime order p, and for Mochizuki's quandle 3-cocycles, which gives the generators of $H^3(R_p, \mathbb{Z}_p) \cong \mathbb{Z}_p$.

4. Quandle cocycle invariant

The quandle cocycle invariant of r-twist-spun $\tau^r K$ of a knot K will be expressed by the Dijkgraaf-Witten invariant of $M_2(K)$ in Corollary 4.3, in the case that the group is $\mathbb{Z}_n$ of odd order n. To show this, first we give the quandle cocycle invariant using the shadow cocycle invariant for tangles, and then using the one for links.

The definition of the shadow cocycle invariant for tangles is similar to the one for links in the previous section. Let T be a tangle of an oriented knot K, and D_T its diagram. Given a quandle coloring C on D_T in a quandle Q, we add the following condition (T) to the definition of shadow colorings of link diagrams.

(T) The color on the unbounded region is the same element with the color on the initial arc, as illustrated in Figure 5.

Now the shadow coloring is uniquely determined by a quandle coloring. For a quandle 3-cocycle ϕ, the shadow cocycle invariant for tangles is the state sum

$$\Psi^*_\phi(K) = \sum_C \prod_x W_\phi(x; C) \in \mathbb{Z}[A],$$

and it does not depend on the choice of a tangle diagram of the knot K [1]. Here the weight $W_\phi(x;C)$ at a crossing x is the same as in Section 3, where S is omitted.

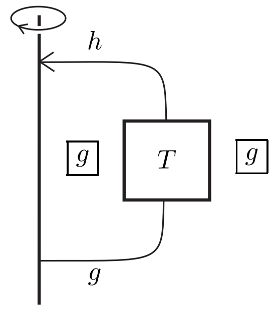


FIGURE 5. The shadow coloring of a tangle diagram

The quandle cocycle invariant $\Phi_\phi(\tau^r K)$ of the r-twist-spun knot K can be computed using a tangle diagram of K. In [1, Lemma 5.2], it is shown that

$$(\sharp) \qquad \Phi_\phi(\tau^r K) = \sum_C \Big[\big\{ \prod_x W_\phi(x;C) \big\}^r \times \big\{ \prod_{k=0}^{r-1} \prod_x W^\sharp_\phi(x; C * h^k) \big\}^{-1} \Big].$$

Here we put $W^\sharp_\phi(x;C)$ to be $\phi(g,g',h)^{\epsilon(x)}$ at each crossing x, where g' is the color on the over-arc, g is the color on its right hand side under-arc, h is the color on the terminal arc of T, and $\epsilon(x)$ is the sign of x determined by the orientations of arcs. The sum is taken for the colorings on T given by the colorings on a diagram of $\tau^r K$, which is obtained by twisting T in $\mathbb{R}^3$.

Proposition 4.1. *Let θ be a group 3-cocycle of an Abelian group G satisfying the conditions (GC2) and (GC3), and $\tilde\theta$ the quandle 3-cocycle given by θ. If r is even, then we have the identity for the quandle cocycle invariant $\Phi_{\tilde\theta}(\tau^r K)$ and the shadow cocycle invariant $\Psi^*_{\tilde\theta}(K)$ for any tangle of a knot K,*

$$\Phi_{\tilde\theta}(\tau^r K) = \rho^r(\Psi^*_{\tilde\theta}(K)).$$

Here the map ρ^r is defined by

$$\rho^r : A \to A, \ t \mapsto t^r.$$

Proof. It is shown in [1, Lemma 5.1] that a coloring C on D_T lifts to a coloring on a diagram of $\tau^r K$, if and only if the identity

$$C(*h)^r = C$$

holds, where h is the color on the terminal arc of T. In our case any coloring on T in Q_G lifts to a coloring on the diagram since

$$(g * h) * h = g$$

for any g and $h \in Q_G$, and r is even.

On the latter term in the sum in ($\sharp$), we have

$$W^{\sharp}_{\tilde\theta}(x; C * h) = W^{\sharp}_{\tilde\theta}(x; C)^{-1}$$

because $\tilde\theta(g * h, g' * h, h) = \tilde\theta(g, g', h)^{-1}$ for any g, g' and $h \in G$. Hence pairs of two weights in the product cancel with each other, and this completes the proof. $\square$

We remark that if r is odd and G is $\mathbb{Z}_n$ of odd order n, then

$$\Phi_\phi(\tau^r K) = n,$$

for any quandle 3-cocycle ϕ, since any coloring on the diagram of $\tau^r K$ in Q_G is trivial.

Proposition 4.2. *Let θ be a group 3-cocycle of $\mathbb{Z}_n$ of odd order n, satisfying the conditions (GC2) and (GC3), and $\tilde\theta$ the quandle 3-cocycle given by θ. We have the identity for the shadow cocycle invariants $\Psi_{\tilde\theta}(L)$ for links and $\Psi^*_{\tilde\theta}(L)$ for their tangles,*

$$\Psi_{\tilde\theta}(L) = |G|\Psi^*_{\tilde\theta}(L).$$

Proof. For any coloring in $\mathbb{Z}_n$ on a diagram D_T of a tangle T, the color on its terminal arc is equal to the one on the initial arc. So the set of colorings on D_T coincides with the set of colorings on D_L of the link L presented by T.

We fixed the shadow coloring of D_T for a quandle coloring C in the condition (T). However, the contribution $\prod_x W_{\tilde\theta}(x, C)$ for all the crossings of D_T does not depend on the shadow coloring, as stated in Section 2, translated in the terms of the Dijkgraaf-Witten invariant. We prove it in a rigorous way here. Prepare two shadow colorings S_1 and S_2 of D_T, associated with the same quandle coloring C. Let the colors around a crossing x as depicted in Figure 6. In this figure the orientations on the arcs are arbitrarily given, and in each region the above (resp. below) element is of S_1 (resp. S_2). Then we have

$$\begin{aligned}\frac{W_{\tilde\theta}(x, C, S_1)}{W_{\tilde\theta}(x, C, S_2)} =& \tilde\theta(s, g, g') \cdot \tilde\theta(s, g, g')^{-1} \\ =& \theta(2g, s-2g, t-s) \cdot \theta(2g, -s, s-t)^{-1} \\ & \cdot\theta(2g', -s, s-t) \cdot \theta(2g', s-2g', t-s)^{-1} \\ & \cdot\theta(-2g+4g', s-2g', t-s) \cdot \theta(-2g+4g', -s+2g-2g', s-t)^{-1} \\ & \cdot\theta(2g', -s+2g-2g', s-t) \cdot \theta(2g', s-2g, t-s)^{-1},\end{aligned}$$

by the group cocycle conditions. Put two terms on each end of the arcs cut around this crossing as illustrated in Figure 6. Then we can see that the terms on the ends coming from its adjacent crossings will cancel with each other. Hence, taking the products for all the crossings of D_T, we have

$$\prod_{x \text{ of } D} W_{\tilde\theta}(x, C, S_1) = \prod_{x \text{ of } D} W_{\tilde\theta}(x, C, S_2).$$

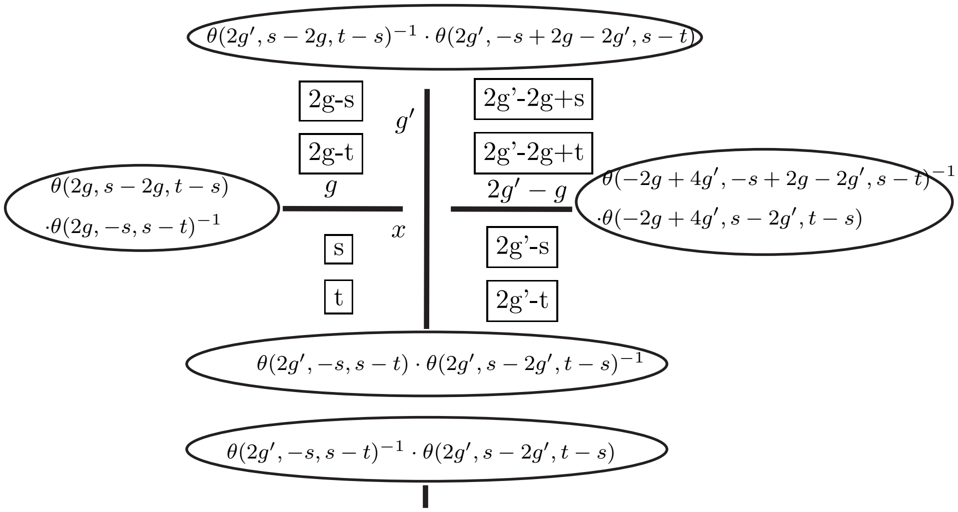


FIGURE 6. Two shadow colorings S_1 (boxed above in the regions) and S_2 (boxed below) associated with the same group coloring

□

Proposition 4.1 is shown in [1] and Proposition 4.2 is shown in [9], both for the dihedral quandle R_p of odd prime order p, and for the Mochizuki's quandle 3-cocycles.

Corollary 4.3. *Let θ be a group 3-cocycle of $\mathbb{Z}_n$ of odd order n, satisfying the conditions (GC2) and (GC3), and $\tilde{\theta}$ the quandle 3-cocycle given by θ. If r is even, then we have the identity between the quandle cocycle invariant $\Phi_{\tilde{\theta}}(\tau^r K)$ of the r-twist-spun $\tau^r K$ of a knot K, and the Dijkgraaf-Witten invariant $Z_\theta(M_2(K))$ of the 2-fold branched covering space $M_2(K)$ branched along K,*

$$\Phi_{\tilde{\theta}}(\tau^r K) = |G|\rho^r(Z_\theta(M_2(K))).$$

The map ρ^r is the one defined in Proposition 4.1.

Proof. By Propositions 4.1 and 4.2, it holds that

$$\Phi_{\tilde{\theta}}(\tau^r K) = \frac{1}{|G|}\rho^r(\Psi_{\tilde{\theta}}(K)).$$

Then Theorem 3.2 leads the identity. □

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